LOWER ALGEBRAIC K-THEORY OF HYPERBOLIC 3-SIMPLEX REFLECTION GROUPS.

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Abstract. A hyperbolic 3-simplex reflection group is a Coxeter group arising as a lattice in $O^+(3,1)$, with fundamental domain a geodesic simplex in $\mathbb{H}^3$ (possibly with some ideal vertices). The classification of these groups is known, and there are exactly 9 cocompact examples, and 23 non-cocompact examples. We provide a complete computation of the lower algebraic K-theory of the integral group ring of all the hyperbolic 3-simplex reflection groups.

1. Introduction

In this paper, we proceed to give a complete computation of the lower algebraic K-theory of the integral group ring of all the hyperbolic 3-simplex reflection groups.

We now proceed to outline the main steps of our approach. Since the groups Γ we are considering are lattices inside $O^+(3,1)$, fundamental results of Farrell and Jones [FJ93] imply that the lower algebraic K-theory of the integral group ring $\mathbb{Z}\Gamma$ can be computed by calculating $H_n^\Gamma(E_{\mathcal{VC}}(\Gamma); \mathbb{K}\mathbb{Z}^{-\infty})$, a specific generalized equivariant homology theory for a model for the classifying space $E_{\mathcal{VC}}(\Gamma)$ of Γ with isotropy in the family $\mathcal{VC}$ of virtually cyclic subgroups of Γ.

After introducing the groups we are interested in (see Section 2), we then combine results from our previous paper [LO] with a recent construction of Lück and Weiermann [LW] to obtain the following explicit formula for the homology group above:

$$K_n(\mathbb{Z}\Gamma) \cong H_n^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{K}\mathbb{Z}^{-\infty}) \oplus \bigoplus_{i=1}^{k} H_n^{V_i}(E_{\mathcal{FIN}}(V_i) \to *).$$

In the formula above, $E_{\mathcal{FIN}}(\Gamma)$ is a model for the classifying space for proper actions, the collection $\{V_i\}_{i=1}^k$ are a finite collection of virtually cyclic subgroups with specific geometric properties, and $H_n^{V_i}(E_{\mathcal{FIN}}(V_i) \to *)$ are cokernels of certain relative assembly maps. This explicit formula is obtained in Section 3.

In view of this explicit formula, our computation reduces to being able to:

(1) identify for each of our groups the corresponding collection $\{V_i\}$ of virtually cyclic subgroups (done in Section 4),
(2) be able to calculate the cokernels of the corresponding relative assembly maps (done in Section 6), and
(3) calculate the homology groups $H_n^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{K}\mathbb{Z}^{-\infty})$.

For the computation of the homology groups, we note that Quinn [Qu82] has developed a spectral sequence for computing the groups $H_n^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{K}\mathbb{Z}^{-\infty})$. The E^2-terms in the spectral sequence can be computed in terms of the lower algebraic K-theory of the stabilizers of cells in a CW-model for the classifying space $E_{\mathcal{FIN}}(\Gamma)$.

In Section 5, we proceed to give, for each of the finite subgroups appearing as a cell stabilizer, a computation of the lower algebraic K-theory. We return to the spectral sequence computation in Section 7, where we analyze some of the maps appearing in the computation of the E^2-terms for the Quinn spectral sequence. In all 32 cases, the spectral sequence collapses at the E^2 stage, allowing us to complete the computations. The reader who is merely interested in knowing the results of the computations is invited to consult Table 6 (for the uniform lattices) and Table 7 (for the non-uniform lattices). Finally, in the Appendix, we provide a "walk through" of the computations for two of the 32 groups we consider.

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2. The Three-dimensional groups

A hyperbolic Coxeter n-simplex Δ^n is an n-dimensional geodesic simplex in $\mathbb{H}^n$, all of whose dihedral angles are submultiples of π or zero. We allow a simplex in $\mathbb{H}^n$ to be unbounded with ideal vertices on the sphere at infinity of $\mathbb{H}^n$. It is known that such simplices exist only in dimensions $n = 2, 3, \ldots, 9$, and that for $n \geq 3$, there are exactly 72 hyperbolic Coxeter simplices up to congruence, see [JKRT99] and [JKRT02].

A hyperbolic Coxeter n-simplex reflection group Γ is the group generated by reflections in the sides of a Coxeter n-simplex in hyperbolic n-space $\mathbb{H}^n$. We will call such group a *hyperbolic n-simplex group.*

According to Vinberg [V67], the associated hyperbolic n-simplex groups of all but eight of the 72 simplices are arithmetic. The nonarithmetic groups are the hyperbolic Coxeter tetrahedra groups $[(3,4,3,5)]$ $[5,3,6]$. $[5,3^{[3]}]$, $[(3^3,6)]$, $[(3,4,3,6)]$, $[(3,5,3,6)]$, and the 5-dimensional hyperbolic Coxeter group $[(3^5,4)]$.

In dimension three, there are 32 hyperbolic Coxeter tetrahedra groups. Nine of them are cocompact, see Figure 1, and 23 are noncocompact, see Figure 2. Let us briefly recall how the algebra and geometry of these groups are encoded in the Coxeter diagrams.

From the algebraic viewpoint, the Coxeter diagram encodes a presentation of the associated group Γ as follows: associate a generator x_i to each vertex v_i of the Coxeter diagram (hence all of our groups will come equipped with four generators, as the Coxeter diagrams have four vertices). For the relations in Γ, one has:

(1) for every vertex v_i, one inserts the relation $x_i^2 = 1$
(2) if two vertices v_i, v_j are not joined by an edge, one inserts the relation $(x_i x_j)^2 = 1$ (so combined with the previous relation, one sees that x_i and x_j commute, generating a $\mathbb{Z}/2 \times \mathbb{Z}/2$),

(3) if two vertices v_i, v_j are joined by an unlabelled edge, one inserts the relation $(x_i x_j)^3 = 1$ (and in particular, the two elements x_i, x_j generate a subgroup isomorphic to the dihedral group D_3),
(4) if two vertices v_i, v_j are joined by an edge with label m_{ij}, one inserts the relation $(x_i x_j)^m_{ij} = 1$ (and hence, the two elements x_i, x_j generate a subgroup isomorphic to the dihedral group $D_{m_{ij}}$).

A *special subgroup* of Γ will be a subgroup generated by a subset of the generating set. Observe that such a subgroup will automatically be a Coxeter group, with a presentation that can again be read off from the Coxeter diagram. Special subgroups generated by a pair of generators will always be isomorphic to a (finite) dihedral group. An important point for our purposes is that in our Coxeter groups, *every* finite subgroup can be conjugated into a finite special subgroup. In particular, since there are only finitely many special subgroups, one can quite easily classify up to isomorphism all the finite subgroups appearing in any of our 32 Coxeter groups.

Now let us move to the geometric viewpoint. As we mentioned earlier, associated to any of our 32 Coxeter groups, one has a simplex Δ^3 in hyperbolic 3-space $\mathbb{H}^3$. Each of the four generators x_i of the Coxeter group Γ is bijectively associated with the hyperplane P_i extending one of the four faces of the simplex Δ^3, and the angles between the respective hyperplanes can again be read off from the Coxeter diagram:

(1) if two vertices v_i, v_j are not joined by an edge, then $\angle(P_i, P_j) = \pi/2$
(2) if two vertices v_i, v_j are joined by an unlabelled edge, then $\angle(P_i, P_j) = \pi/3$
(3) if two vertices v_i, v_j are joined by an edge with label m_{ij}, then $\angle(P_i, P_j) = \pi/m_{ij}$.

The resulting configuration of four hyperplanes exists, and is unique up to isometries of H^3. One can now define the map $\Gamma \to O^+(3,1) = Isom(\mathbb{H}^3)$ by sending each generator x_i to the isometry obtained by reflecting in the corresponding hyperplane P_i. The condition on the angles between the hyperplanes ensures that this map respects the relations in Γ, and hence is actually a homomorphism. In fact this map is an embedding of Γ as a discrete subgroup of $O^+(3,1)$, with fundamental domain for the associated action on $\mathbb{H}^3$ consisting precisely of the simplex Δ^3.

Finally, to relate the geometric with the algebraic viewpoint, we remind the reader of the following bijective identifications:

(1) given an edge in the 3-simplex Δ^3, lying on the intersection of two hyperplanes P_i, P_j, the subgroup of Γ that fixes the edge pointwise is precisely the special subgroup $\langle x_i, x_j \rangle$ (and hence will be a dihedral group).
(2) given a vertex in the 3-simplex Δ^3, obtained as the intersection of three hyperplanes P_i, P_j, P_k, the subgroup of Γ that stabilizes the vertex is precisely the special subgroup $\langle x_i, x_j, x_k \rangle$.

We point out that the stabilizer of a vertex of Δ^3 will either be a finite Coxeter group (if the vertex lies inside $\mathbb{H}^3$), or will be a 2-dimensional crystallographic group (if the vertex is an ideal vertex). Furthermore, one can readily determine whether a vertex will be ideal or not, just by determining whether the associated special subgroup is crystallographic or finite.

It is known that for all the groups listed above the Farrell and Jones Isomorphism Conjecture in lower algebraic K-theory holds, that is $H_n^\Gamma(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong$

$K_n(\mathbb{Z}\Gamma)$ for $n < 2$ (see [Or04, Theorem 2.1]). Our plan is to use this result to explicitly compute the lower algebraic K-theory of the integral group ring $\mathbb{Z}\Gamma$, for all of the 32 groups listed above.

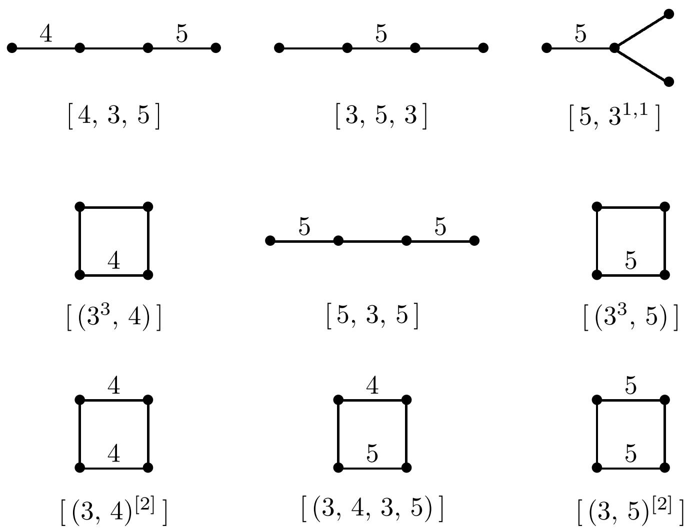

FIGURE 1. Cocompact hyperbolic Coxeter tetrahedral groups

3. A formula for the algebraic K-theory.

In this section, we combine some recent work of Lück and Weiermann [LW] with some previous work of the authors [LO] to establish the following:

Proposition 3.1. *Let $\mathcal{F} \subset \widetilde{\mathcal{F}}$ be a nested pair of families of subgroups of Γ, and assume that the collection of subgroups $\{H_\alpha\}_{\alpha\in I}$ is adapted to the pair $(\mathcal{F}, \widetilde{\mathcal{F}})$. Let $\mathcal{H}$ be a complete set of representatives of the conjugacy classes within $\{H_\alpha\}$, and consider the cellular Γ-pushouts:*

$$\begin{tikzcd} \coprod_{H\in\mathcal{H}} \Gamma \times_H E_{\mathcal{F}}(H) \arrow[r, "\beta"] \arrow[d, "\alpha"'] & E_{\mathcal{F}}(\Gamma) \arrow[d] \\ \coprod_{H\in\mathcal{H}} \Gamma \times_H E_{\widetilde{\mathcal{F}}}(H) \arrow[r] & X \end{tikzcd}$$

Then X is a model for $E_{\widetilde{\mathcal{F}}}(\Gamma)$. In the above cellular Γ-pushout, we require either (1) α is the disjoint union of cellular H-maps ($H \in \mathcal{H}$), β is an inclusion of Γ-CW-complexes, or (2) α is the disjoint union of inclusions of H-CW-complexes ($H \in \mathcal{H}$), β is a cellular Γ-map.

Proof. Let us start by recalling that a collection $\{H_\alpha\}_{\alpha\in I}$ of subgroups of Γ is adapted to the pair $(\mathcal{F}, \widetilde{\mathcal{F}})$ provided that:

(1) For all $H_1, H_2 \in \{H_\alpha\}_{\alpha\in I}$, either $H_1 = H_2$, or $H_1 \cap H_2 \in \mathcal{F}$.

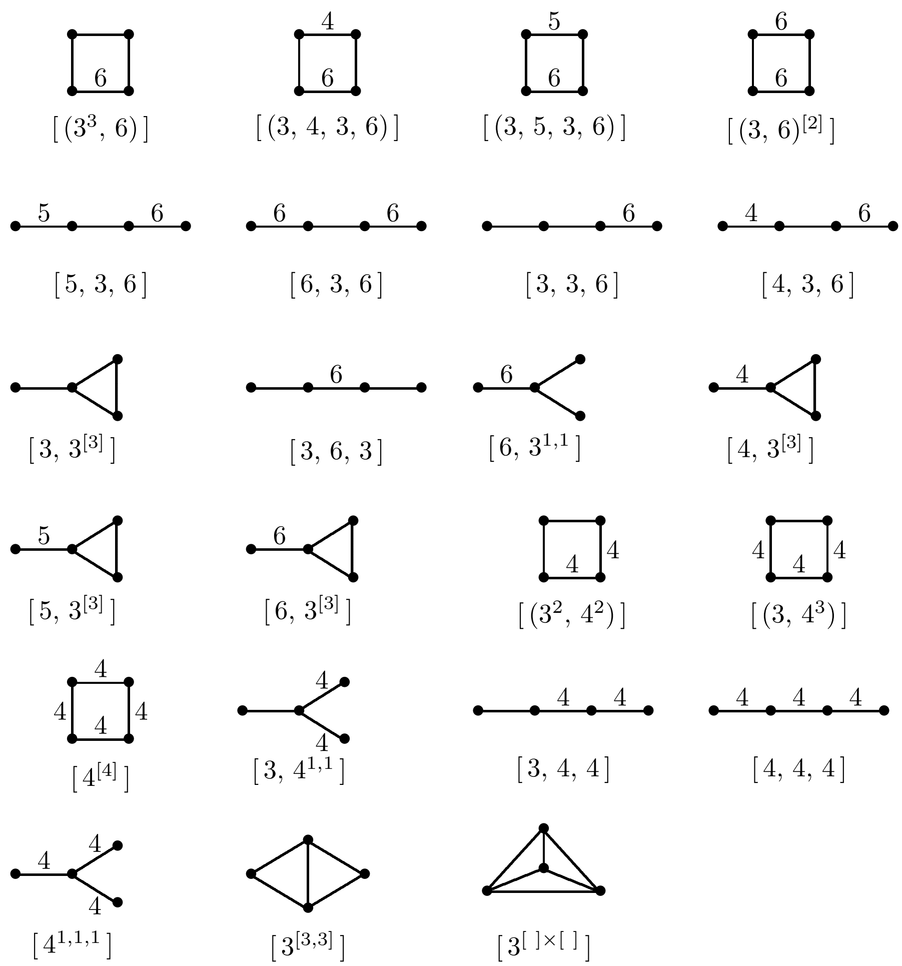

FIGURE 2. Noncocompact hyperbolic Coxeter tetrahedral groups

(2) The collection $\{H_\alpha\}_{\alpha\in I}$ is *conjugacy closed* i.e. if $H \in \{H_\alpha\}_{\alpha\in I}$ then $gHg^{-1} \in \{H_\alpha\}_{\alpha\in I}$ for all $g \in \Gamma$.
(3) Every $H \in \{H_\alpha\}_{\alpha\in I}$ is *self-normalizing*, i.e. $N_\Gamma(H) = H$.
(4) For all $G \in \widetilde{\mathcal{F}} \setminus \mathcal{F}$, there exists $H \in \{H_\alpha\}_{\alpha\in I}$ such that $G \leq H$.

Note that the subgroups in the collection $\{H_\alpha\}_{\alpha\in I}$ are *not* assumed to lie within the family $\widetilde{\mathcal{F}}$.

Using the existence of the adapted family $\{H_\alpha\}_{\alpha\in I}$, one can now define an equivalence relation on the subgroups in $\widetilde{\mathcal{F}} - \mathcal{F}$ as follows: we decree that $G_1 \sim G_2$ if there exists an $H \in \{H_\alpha\}_{\alpha\in I}$ such that $G_1 \leq H$ and $G_2 \leq H$. Note that $\sim$ is indeed an equivalence relation: the symmetric property is immediate, while reflexivity follows from property (4) of adapted collection, and transitivity comes from property (1) of adapted collection. Furthermore this equivalence relation has the following two properties:

- if $G_1, G_2 \in \widetilde{\mathcal{F}} - \mathcal{F}$ satisfies $G_1 \leq G_2$, then $G_1 \sim G_2$ (immediate from the definition of $\sim$).
- if $G_1, G_2 \in \widetilde{\mathcal{F}} - \mathcal{F}$ and $g \in \Gamma$, then $G_1 \sim G_2 \Leftrightarrow gG_1g^{-1} \sim gG_2g^{-1}$ (follows from property (2) of adapted collection).

We denote by $[\widetilde{\mathcal{F}} - \mathcal{F}]$ the set of equivalence classes of elements in $\widetilde{\mathcal{F}} - \mathcal{F}$ under the above equivalence relation, and for $G \in \widetilde{\mathcal{F}} - \mathcal{F}$, we will write $[G]$ for the corresponding equivalence class. Note that by the second property above, the Γ-action by conjugation on $\widetilde{\mathcal{F}} - \mathcal{F}$ preserves equivalence classes, and hence descends to a Γ-action on $[\widetilde{\mathcal{F}} - \mathcal{F}]$. We let I be a complete set of representatives $[G]$ of the Γ-orbits in $[\widetilde{\mathcal{F}} - \mathcal{F}]$. Finally, we define for $G \in \widetilde{\mathcal{F}} - \mathcal{F}$ the subgroup:

$$N_\Gamma[G] := \{g \in \Gamma \mid [gGg^{-1}] = [G]\}$$

which is precisely the isotropy group of $[G] \in [\widetilde{\mathcal{F}} - \mathcal{F}]$ under the Γ-action induced by conjugation. Finally, define a family of subgroups $\widetilde{\mathcal{F}}[G]$ of the group $N_\Gamma[G]$ by:

$$\widetilde{\mathcal{F}}[G] := \{K \subset N_\Gamma[G] \mid K \in \widetilde{\mathcal{F}} - \mathcal{F}, [K] = [G]\} \ \cup \ \{K \subset N_\Gamma[G] \mid K \in \mathcal{F}\}$$

Observe that the notions defined above (introduced in [LW]) make sense for *any* equivalence relation on $\widetilde{\mathcal{F}} - \mathcal{F}$ satisfying the two properties above.

Now [LW, Theorem 2.3] states that for any equivalence relation $\sim$ on the elements in $\widetilde{\mathcal{F}} - \mathcal{F}$ satisfying the two properties above (and with the notation used in the previous paragraph), the Γ-CW-complex X defined by the cellular Γ-pushout depicted below is a model for $E_{\widetilde{\mathcal{F}}}(\Gamma)$.

$$\begin{tikzcd}
\coprod_{[H]\in I} \Gamma \times_{N_\Gamma[H]} E_{\mathcal{F}\cap N_\Gamma[H]}(N_\Gamma[H]) \arrow[r, "\beta"] \arrow[d, "\alpha"'] & E_{\mathcal{F}}(\Gamma) \arrow[d] \\
\coprod_{[H]\in I} \Gamma \times_{N_\Gamma[H]} E_{\widetilde{\mathcal{F}}[H]}(N_\Gamma[H]) \arrow[r] & X
\end{tikzcd}$$

In the above cellular Γ-pushout, Lück-Weiermann require either (1) α is the disjoint union of cellular $N_\Gamma[H]$-maps ($[H] \in I$), β is an inclusion of Γ-CW-complexes, or (2) α is the disjoint union of inclusions of $N_\Gamma[H]$-CW-complexes ($[H] \in I$), β is a cellular Γ-map.

We now proceed to verify that, for the equivalence relation we have defined using the adapted family $\{H_\alpha\}_{\alpha \in I}$, the left hand terms in the cellular Γ-pushout given above reduce to precisely the left hand terms appearing in the statement of our proposition. This boils down to two claims:

Claim 1: For any $G \in \widetilde{\mathcal{F}} - \mathcal{F}$, we have the equality $N_\Gamma[G] = H$ where H is the unique element in $\{H_\alpha\}_{\alpha \in I}$ satisfying $G \leq H$.

To see this, we first note that there indeed is a unique $H \in \{H_\alpha\}_{\alpha \in I}$ satisfying $G \leq H$, for if there were two such groups $H_1 \neq H_2$, then we would immediately see that $H_1 \cap H_2 \geq G \in \widetilde{\mathcal{F}} - \mathcal{F}$, contradicting the property (1) of an adapted collection. Next we observe that if $h \in H$, then $hGh^{-1} \leq hHh^{-1} = H$, and hence that $[hGh^{-1}] = [G]$, which implies the containment $H \leq N_\Gamma[G]$. Conversely, if $k \in N_\Gamma[G]$, then we have that $[G] = [kGk^{-1}]$, and so from the definition of the equivalence relation there must exist some $\bar{H} \in \{H_\alpha\}_{\alpha \in I}$ with $G \leq \bar{H}$ and $kGk^{-1} \leq \bar{H}$. Since we already know that $G \leq H$, the uniqueness forces $\bar{H} = H$,

and thus that $kGk^{-1} \leq H$. This in turn tells us that $H \cap k^{-1}Hk \geq G \in \widetilde{\mathcal{F}} - \mathcal{F}$, and property (1) of an adapted collection now forces $H = k^{-1}Hk$, which implies that $k \in N_\Gamma(H)$. But property (3) of an adapted collection forces the group H to be self-normalizing, giving $k \in H$, and completing the proof of the reverse inclusion.

Claim 2: For any $G \in \widetilde{\mathcal{F}} - \mathcal{F}$, the family $\widetilde{\mathcal{F}}[G]$ on the group $N_\Gamma[G] = H$ (see the previous Claim) coincides with the restriction $\widetilde{\mathcal{F}} \cap H$ of the family $\widetilde{\mathcal{F}}$ to the subgroup H (i.e. consisting of all elements in $\widetilde{\mathcal{F}}$ that lie within H).

Note that the containment $\widetilde{\mathcal{F}}[G] \subset \widetilde{\mathcal{F}} \cap H$ is obvious from the definition of $\widetilde{\mathcal{F}}[G]$. For the opposite containment, let $K \in \widetilde{\mathcal{F}} \cap H \subset \widetilde{\mathcal{F}}$, and observe that $K \leq H$ and either $K \in \mathcal{F}$, or $K \in \widetilde{\mathcal{F}} - \mathcal{F}$. In the first case, we have $K \in \mathcal{F} \cap H \subset \widetilde{\mathcal{F}}[G]$, while in the second case, we have that $[K] = [G]$ by the definition of the equivalence relation, and hence again we have $K \in \widetilde{\mathcal{F}}[G]$. This gives us the containment $\widetilde{\mathcal{F}} \cap H \subset \widetilde{\mathcal{F}}[G]$, giving us the Claim.

Having established our two Claims, we can now substitute the expressions from the Claims for the corresponding ones in the Lück-Weiermann diagram. Finally, we comment on the indices in the disjoint sums appearing in the right hand of the diagrams. In the expression of Lück-Weiermann, the disjoint sum is taken over I, a complete system of representatives $[G]$ of the Γ-orbits in $[\widetilde{\mathcal{F}} - \mathcal{F}]$. But observe that from the definition of the equivalence relation we are using, classes in $[\widetilde{\mathcal{F}} - \mathcal{F}]$ can be bijectively identified with groups $H \in \{H_\alpha\}_{\alpha \in I}$ (by associating each class in $[\widetilde{\mathcal{F}} - \mathcal{F}]$ with the unique element in $\{H_\alpha\}_{\alpha \in I}$ containing all the elements in the class). Since it is clear that the Γ-action on $[\widetilde{\mathcal{F}} - \mathcal{F}]$ coincides (under the bijection above) with the Γ-action on the set $\{H_\alpha\}_{\alpha \in I}$, we can replace the system of representatives I by the system of representatives $\mathcal{H}$. This completes the proof of the proposition. □

We now specialize to the case where $\mathcal{F} = \mathcal{FIN}$ and $\widetilde{\mathcal{F}} = \mathcal{VC}$, and recall that Bartels [Bar03] has established that for any group Γ, the relative assembly map:

$$H_*^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \to H_*^\Gamma(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty})$$

is split injective. We denote by $H_*^\Gamma(E_{\mathcal{FIN}}(\Gamma) \to E_{\mathcal{VC}}(\Gamma))$ the cokernel of the relative assembly map. Now applying the induction structure on this equivariant generalized homology theory, an immediate consequence of the previous proposition is the following:

Corollary 3.2. *Given the group Γ, assume that the collection of subgroups $\{H_\alpha\}_{\alpha \in I}$ is adapted to the pair $(\mathcal{FIN}, \mathcal{VC})$. If $\mathcal{H}$ be a complete set of representatives of the conjugacy classes within $\{H_\alpha\}$, then we have a splitting:*

$$H_*^\Gamma(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong H_*^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{H \in \mathcal{H}} H_*^H(E_{\mathcal{FIN}}(H) \to E_{\mathcal{VC}}(H)).$$

Next we recall that the authors established in [LO, Theorem 2.6] that in the case where Γ is hyperbolic relative to a collection of subgroups $\{H_i\}_{i=1}^k$ (assumed to be pairwise non-conjugate), then the collection of subgroups consisting of:

(1) All conjugates of H_i (these will be called *peripheral subgroups*).
(2) All maximal virtually infinite cyclic subgroups V such that $V \nsubseteq gH_ig^{-1}$, for all $i = 1, \cdots k$, and for all $g \in \Gamma$.

is adapted to the pair of families $(\mathcal{FIN}, \mathcal{VC})$. Applying the previous corollary to this special case, we get:

Corollary 3.3. *Assume that the group Γ is hyperbolic relative to the collection of subgroups $\{H_i\}_{i=1}^k$ (assumed to be pairwise nonconjugate). Let $\mathcal{V}$ be a complete set of representatives of the conjugacy classes of maximal virtually infinite cyclic subgroups inside Γ which cannot be conjugated within any of the H_i. Then we have a splitting:*

$$H_*^\Gamma(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong H_*^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{i=1}^k H_*^{H_i}(E_{\mathcal{FIN}}(H_i) \to E_{\mathcal{VC}}(H_i))$$

$$\oplus \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{FIN}}(V) \to *).$$

The primary example of relatively hyperbolic groups are groups Γ acting with cofinite volume (but not cocompactly) on a simply connected Reimannian manifold whose sectional curvature satisfies $-b^2 \leq K \leq -a^2 < 0$. These groups are hyperbolic relative to the "cusp groups", which one can take to be the subgroups arising as stabilizers of ideal points in the boundary at infinity of the Riemannian manifold. We note that non-uniform lattices in $O^+(n,1) = Isom(\mathbb{H}^n)$ are examples of relatively hyperbolic groups, and for this class of groups, the cusp groups are automatically $(n-1)$-dimensional crystallographic groups (this is due to the fact that the horospheres have intrinsic geometry isometric to $\mathbb{R}^{n-1}$). Observe that 23 of the groups we are considering (see Figure 2) are non-uniform lattices in $O^+(3,1)$, and hence are relatively hyperbolic groups, relative to a collection of subgroups, each of which is isomorphic to a 2-dimensional crystallographic group. In the situation of the groups we are considering, the situation is even further simplified by the following observations:

- Pearson [Pe98] showed that for *any* 2-dimensional crystallographic group H, the relative assembly map is an isomorphism for $n \leq 1$, and hence that:
$$H_n^H(E_{\mathcal{FIN}}(H) \to E_{\mathcal{VC}}(H)) = 0$$
for $n \leq 1$.
- the authors in [LO, Section 3] gave a general procedure for classifying the maximal virtually cyclic subgroups of Coxeter groups acting on $\mathbb{H}^3$. The groups fall into three types, with infinitely many conjugacy classes of type II and type III, and only *finitely many* conjugacy classes of type I subgroups. Furthermore, the relative assembly map is an isomorphism (for $n \leq 1$) for all groups of type II and III. This is discussed in more detail in Section 4.
- Berkove, Farrell, Juan-Pineda, and Pearson [BFPP00] established that the Farrell-Jones isomorphism conjecture holds for all lattices Γ in hyperbolic space (and $k \leq 1$), and hence that one has isomorphisms:
$$K_n(\mathbb{Z}\Gamma) \cong H_n^\Gamma(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty})$$
for all $n \leq 1$.

Combining these observations with the previous corollary yields the following:

Corollary 3.4. *Let $\Gamma \leq O^+(3,1)$ be any Coxeter group arising as a lattice (uniform or non-uniform), and let $\{V_i\}_{i=1}^k$ be a complete set of representatives for conjugacy*

classes of type I maximal virtually infinite cyclic subgroups of Γ*. Then we have, for all* $n \leq 1$*, isomorphisms:*

$$K_n(\mathbb{Z}\,\Gamma) \cong H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{i=1}^{k} H_n^{V_i}(E_{\mathcal{FIN}}(V_i) \to *).$$

In the next sections, we will implement this corollary to compute the lower algebraic K-theory of the integral group rings of all 32 of the 3-simplex hyperbolic reflection groups.

4. Maximal infinite $\mathcal{VC}_\infty$ subgroups.

In this section, we proceed to classify the maximal infinite virtually cyclic ($\mathcal{VC}_\infty$) subgroups arising in our groups. Let us start by briefly recalling some of the results from Section 3 of [LO]. First of all, for a lattice Γ in $O^+(n,1)$, infinite $\mathcal{VC}$ subgroups are of two types: those that fix a single point in the boundary at infinity, and those that fix a pair of points in the boundary at infinity. We call subgroups of the first type *parabolic*, and those of second type *hyperbolic*. Note that every parabolic subgroup can be conjugated into a cusp group; for the purpose of our classification, we will ignore these subgroups. The subgroups of hyperbolic type automatically stabilize the geodesic joining the pair of fixed points in the boundary at infinity. Furthermore the geodesic they stabilize will project to a periodic curve in the quotient space $\mathbb{H}^n/\,\Gamma$. Note that conversely, stabilizers of periodic geodesics are infinite $\mathcal{VC}$ subgroups of Γ. This implies that the *maximal* hyperbolic type infinite $\mathcal{VC}$ subgroups of Γ are in bijective correspondence with stabilizers of periodic geodesics.

We now specialize to the case where $n = 3$, and Γ is a Coxeter group. In this situation, we can subdivide the family of periodic geodesics into three types:

- a geodesic whose projection has non-trivial intersection with the interior of the polyhedron $\mathbb{H}^3/\,\Gamma$, which we call type III.
- a geodesic whose projection lies in the boundary of the polyhedron $\mathbb{H}^3/\,\Gamma$, but does not lie inside the 1-skeleton of $\mathbb{H}^3/\,\Gamma$, which we call type II.
- a geodesic whose projection lies in the 1-skeleton of the polyhedron $\mathbb{H}^3/\,\Gamma$, which we call type I.

For geodesics of type III, it is easy to see that the stabilizer of the geodesic must be isomorphic to either $\mathbb{Z}$ or D_∞. For geodesics of type II, the stabilizer is always isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}$ or $\mathbb{Z}_2 \times D_\infty$. The main purpose of this section will be to classify stabilizers of type I geodesics for all 32 groups which occur as hyperbolic 3-simplex reflection groups.

We start by outlining our approach: all the groups we are considering have fundamental domain consisting of a 3-dimensional simplex in $\mathbb{H}^3$ (possibly with some ideal vertices). So up to conjugacy, for each of the groups we are considering, we can have *at most six* distinct stabilizers of type I geodesic (one for each edge in the fundamental domain, fewer in the presence of ideal vertices). But this is actually an overcount, as one could potentially have a type I geodesic whose projection into the fundamental domain passes through several of the edges. So the first step is to understand *how many* distinct stabilizers (up to conjugacy) one obtains.

Let us explain how one can find out the number of distinct stabilizers. Note that, at every (non-ideal) vertex v of our fundamental domain 3-simplex in $\mathbb{H}^3$, we can consider a small ϵ-sphere S_v centered at v. Now the tessellation of $\mathbb{H}^3$ by

copies of the fundamental domain induces a tessellation of S_v by isometric spherical triangles. In fact, the tessellation of S_v is the one naturally associated with the special subgroup of the Coxeter group Γ that stabilizes the vertex v. Now note that, if we were to label the three edges of the 3-simplex incident to v, we get a corresponding label of the three vertices of a spherical triangle in the tessellation of S_v. One can extend this labeling via reflections, both for the tessellation of $\mathbb{H}^3$ and the tessellation of S_v.

Now given a periodic geodesic of type I, with a portion of the geodesic projecting to the edge e in the 3-simplex, with e adjacent to the vertex v, one can easily "read off" from the labeled tessellation of S_v which edge extends the geodesic. Indeed, this will be picked up by the label of the vertex in the tessellation of S_v which is antipodal to the labeled vertex corresponding to e. In this manner, one can easily decide the number of distinct stabilizers of type I geodesics that arise for the 32 groups we are considering.

To recognize the tessellations arising for the various S_v, one now notes that the isometry group of each of these tessellations can be obtained by looking at the stabilizer Γ_v of the vertex v in the group Γ. These stabilizers are finite special subgroups, of the Coxeter group Γ, generated by three of the four canonical generators of Γ. From the classification of the 32 hyperbolic 3-simplex groups, it is easy to list out all the finite parabolic subgroups we need to consider: there are eight of these, namely $\mathbb{Z}_2 \times D_2$, $\mathbb{Z}_2 \times D_3$, $\mathbb{Z}_2 \times D_4$, $\mathbb{Z}_2 \times D_5$, $\mathbb{Z}_2 \times D_6$, $[3,3] \cong S_4$, $[3,4] \cong \mathbb{Z}_2 \times S_4$, and $[3,5] \cong \mathbb{Z}_2 \times A_5$.

The next step is to identify the stabilizers of the corresponding geodesics. In the situation we are considering, all the type I geodesics η that appear have stabilizer $\mathrm{Stab}_\Gamma(\eta)$ acting with fundamental domain an interval. In fact, the interval can be identified with the quotient space $\eta/\,\mathrm{Stab}_\Gamma(\eta) \subset \mathbb{H}^3/\,\Gamma$, which will be a union of edges in the 1-skeleton of the 3-simplex $\mathbb{H}^3/\,\Gamma$. Hence the group $\mathrm{Stab}_\Gamma(\eta)$ can be identified using Bass-Serre theory: it will be the fundamental group of a graph of group, where the graph of group consists of a single edge joining two vertices, with edge/vertex groups which can be explicitly found from the tessellations. We will say that the geodesic (or sometimes the edge in the 1-skeleton) *reflects* at the two endpoint vertices.

Indeed, the edge group G_e will be precisely the stabilizer of one of the edges in $\eta/\,\mathrm{Stab}_\Gamma(\eta) \subset \mathbb{H}^3/\,\Gamma$. On the other hand, the vertex groups G_v, G_w can be found by looking at each of the two endpoint vertices v, w for $\eta/\,\mathrm{Stab}_\Gamma(\eta)$, and studying the spherical tessellations of S_v, S_w. Note that we are trying to identify elements in Γ, which stabilize the vertex v (respectively w), and *additionally* map the geodesic η through v to itself. In particular, it must map the pair of antipodal vertices $\eta^\pm$ (corresponding to the incoming/outgoing η-directions) in the tessellation of S_v to themselves. The subgroup $G_e \subset G_v$ can be identified with the index 2 subgroup consisting of elements G_v which fix both of the points $\eta^\pm$. Now there is an obvious map which permutes the two points η^+ and η^-: namely the reflection in the equator equidistant from these two points. But it is not clear that this reflection preserves the tessellation of S_v; in some cases, one will need to reflect in the equator, and then rotate by a certain angle along the $\eta^\pm$ axis, in order to obtain an element in Γ_v. Note that if the reflection in the equator *preserves* the tessellation, then we immediately obtain that $G_v \cong G_e \times \mathbb{Z}_2$. If the reflection in the equator *does not preserve* the tessellation, then we obtain that $G_v \cong G_e \rtimes \mathbb{Z}_2$. One can perform

the same analysis at the vertex w, and hence find an expression for $\mathrm{Stab}_\Gamma(\eta)$ as an amalgamation of the groups G_v, G_w over the index 2 subgroups G_e.

We make two observations: first of all, the stabilizer of an edge will always be a special subgroup of Γ, generated by a pair of canonical generators in Γ. In particular, the group G_e will always be a dihedral group D_k for some k. Now the vertex groups are of two types: (1) if the reflection in the equator preserves the tessellation, we obtain $G_v \cong \mathbb{Z}_2 \times D_k$, or (2) if the reflection in the equator does not preserve the tessellation, then one can explicitly read off the semi-direct product structure from the tessellation, and in fact it is easy to see that $G_v \cong D_{2k}$. In the table below, we list out, for each of the finite parabolic subgroups we need to consider, the edges that reflect, as well as the corresponding G_v. Let us explain the notation used in the table: the first column gives the various finite parabolic subgroups that occur, the second column lists the angles that appear in the spherical triangles of the corresponding tessellation of S_v. The remaining three columns are ordered from smallest angle to largest, and expresses whether (1) the corresponding edge extends (i.e. does not reflect) at v, and (2) if it reflects, the corresponding subgroup G_v.

$\mathbb{Z}_2 \times D_2$	$\pi/2, \pi/2, \pi/2$	$\mathbb{Z}_2 \times D_2$	$\mathbb{Z}_2 \times D_2$	$\mathbb{Z}_2 \times D_2$
$\mathbb{Z}_2 \times D_3$	$\pi/3, \pi/2, \pi/2$	$\mathbb{Z}_2 \times D_3$	extends	extends
$\mathbb{Z}_2 \times D_4$	$\pi/4, \pi/2, \pi/2$	$\mathbb{Z}_2 \times D_4$	extends	extends
$\mathbb{Z}_2 \times D_5$	$\pi/5, \pi/2, \pi/2$	$\mathbb{Z}_2 \times D_5$	extends	extends
$\mathbb{Z}_2 \times D_6$	$\pi/6, \pi/2, \pi/2$	$\mathbb{Z}_2 \times D_6$	extends	extends
S_4	$\pi/3, \pi/3, \pi/2$	extends	extends	D_4
$\mathbb{Z}_2 \times S_4$	$\pi/4, \pi/3, \pi/2$	$\mathbb{Z}_2 \times D_4$	D_6	$\mathbb{Z}_2 \times D_2$
$\mathbb{Z}_2 \times A_5$	$\pi/5, \pi/3, \pi/2$	D_{10}	D_6	$\mathbb{Z}_2 \times D_2$

Table 1: Finite parabolic subgroups & local behavior of edges.

From the Coxeter diagrams of the 32 groups we are considering, we can now read off quite easily the number (up to conjugacy) of stabilizers of type I geodesics. We now proceed to summarize the results of this procedure, which we list out in Tables 2, 3, and 4. We remind the reader that, in addition to these subgroups, there will also be (up to conjugacy) countably infinitely many maximal $\mathcal{VC}$ subgroup of hyperbolic type isomorphic to one of $\mathbb{Z}, D_\infty, \mathbb{Z}_2 \times \mathbb{Z}, \mathbb{Z}_2 \times D_\infty$ (coming from stabilizers of type II and type III geodesics). The list below can be thought of as the "exceptional" maximal $\mathcal{VC}$ subgroups of hyperbolic type. Indeed, as we will see in the subsequent sections, these will be the only maximal $\mathcal{VC}$ subgroups of hyperbolic type that will actually contribute to the algebraic K-theory of the ambient groups.

4.1. **The uniform lattices.** There are 9 hyperbolic 3-simplex groups with fundamental domain a compact 3-simplex in $\mathbb{H}^3$. The number and type of (non-finite) stabilizers of type I geodesics are listed in the following table:

Γ	Stab_Γ
$[(3^3,5)]$	$D_6 *_{D_3} D_6, (D_2 \times \mathbb{Z}_2) *_{D_2} D_4$ (twice)$, D_{10} *_{D_5} D_{10}$
$[5,3,5]$	$D_2 \times D_\infty, D_6 *_{D_3} D_6, (D_5 \times \mathbb{Z}_2) *_{D_5} D_{10}$ (twice)
$[(3^3,4)]$	$D_4 \times D_\infty, D_6 *_{D_3} D_6, (D_2 \times \mathbb{Z}_2) *_{D_2} D_4$ (twice)
$[3,5,3]$	$D_2 \times D_\infty, (D_3 \times \mathbb{Z}_2) *_{D_3} D_6$(twice)$, D_{10} *_{D_5} D_{10}$
$[5,3^{1,1}]$	$D_2 \times D_\infty$ (twice)$, D_6 *_{D_3} D_6, D_{10} *_{D_5} D_{10}, (D_2 \times \mathbb{Z}_2) *_{D_2} D_4$
$[4,3,5]$	$D_2 \times D_\infty,$ (twice)$, D_4 \times D_\infty, D_6 *_{D_3} D_6, (D_5 \times \mathbb{Z}_2) *_{D_5} D_{10}$
$[(3,5)^{[2]}]$	$D_2 \times D_\infty$ (twice)$, D_6 *_{D_3} D_6$(twice)$, D_{10} *_{D_5} D_{10}$(twice)
$[(3,4,3,5)]$	$D_2 \times D_\infty$ (twice)$, D_4 \times D_\infty, D_6 *_{D_3} D_6$(twice)$, D_{10} *_{D_5} D_{10}$
$[(3,4)^{[2]}$	$D_2 \times D_\infty$ (twice)$, D_4 \times D_\infty$(twice)$, D_6 *_{D_3} D_6$(twice)

Table 2: Structure of subgroups of cocompact groups.

4.2. **One ideal vertex.** We have nine such Coxeter groups, namely the groups: $[5,3^{[3]}]$, $[5,3,6]$, $[3^2,4^2]$, $[4,3^{[3]}]$, $[3,3^{[3]}]$, $[3,4^{1,1}]$, $[4,3,6]$, $[3,3,6]$, and $[3,4,4]$. The number and type of (non-finite) stabilizers of type I geodesics, as well as the cusp subgroups are listed in the following table:

Γ		Stab_Γ	Cusp
$[3,3^{[3]}]$	1 edge	$D_4 *_{D_2} D_4$	$[3^{[3]}]$
$[3,3,6]$	1 edge	$(D_2 \times \mathbb{Z}_2) *_{D_2} D_4$	$[3,6]$
$[5,3^{[3]}]$	2 edges	$D_2 \times D_\infty, \quad D_{10} *_{D_5} D_{10}$	$[3^{[3]}]$
$[5,3,6]$	2 edges	$D_2 \times D_\infty, \quad (D_5 \times \mathbb{Z}_2) *_{D_5} D_{10}$	$[3,6]$
$[(3^2,4^2)]$	2 edges	$D_2 \times D_\infty, \quad D_6 *_{D_3} D_6$	$[4,4]$
$[4,3^{[3]}]$	2 edges	$D_2 \times D_\infty, \quad D_4 \times D_\infty$	$[3^{[3]}]$
$[3,4,4]$	2 edges	$D_2 \times D_\infty, \quad (D_3 \times \mathbb{Z}_2) *_{D_3} D_6$	$[4,4]$
$[3,4^{1,1}]$	3 edges	$D_2 \times D_\infty$(twice)$, \quad D_6 *_{D_3} D_6$	$[4,4]$
$[4,3,6]$	3 edges	$D_2 \times D_\infty$(twice)$, \quad D_4 \times D_\infty$	$[3,6]$

Table 3: Structure of subgroups of 1-ideal vertex groups.

4.3. **Two ideal vertices.** We have nine such Coxeter groups, namely the groups: $[(3,5,3,6)]$, $[(3,4^3)]$, $[(3,4,3,6)]$, $[(3^3,6)]$, $[3^{[3,3]}]$, $[6,3^{1,1}]$, $[3,6,3]$, $[6,3,6]$, and $[4,4,4]$. Note that for these groups, we have only one edge segment in the fundamental domain to consider.

For the groups $[(3^3, 6)]$ and $[3, 6, 3]$, the edge extends to one of the non-compact edges, and hence we again have that there are no periodic geodesics of type I. So in both of these cases, we have that the maximal virtually infinite $\mathcal{VC}$ subgroups of hyperbolic type are isomorphic to $\mathbb{Z}$, D_∞, $\mathbb{Z}_2 \times \mathbb{Z}$, or $\mathbb{Z}_2 \times D_\infty$.

In the remaining cases, the edge reflects at both of its endpoints. The stabilizers we obtain, as well as the two cusp subgroups, are listed out in the following table:

Γ	Stab_Γ	Cusp
$[(3,5,3,6)]$	$D_{10} *_{D_5} D_{10}$	$[3,6]$ (twice)
$[(3,4^3)]$	$D_6 *_{D_3} D_6$	$[4,4]$ (twice)
$[(3,4,3,6)]$	$D_4 \times D_\infty$	$[3,6]$ (twice)
$[3^{[3,3]}]$	$D_4 *_{D_2} D_4$	$[3^{[3]}]$ (twice)
$[6,3^{1,1}]$	$(D_2 \times \mathbb{Z}_2) *_{D_2} D_4$	$[3,6]$ (twice)
$[6,3,6]$	$D_3 \times D_\infty$	$[3,6]$ (twice)
$[4,4,4]$	$D_2 \times D_\infty$	$[4,4]$ (twice)

Table 4: Structure of subgroups of 2-ideal vertex groups.

4.4. **Three ideal vertices.** We have two such Coxeter group: $[6, 3^{[3]}]$ and $[4^{1,1,1}]$. Again, there will be no periodic geodesics of type I. So we obtain that the maximal virtually infinite $\mathcal{VC}$ subgroups of hyperbolic type are isomorphic to $\mathbb{Z}$, D_∞, $\mathbb{Z}_2 \times \mathbb{Z}$, or $\mathbb{Z}_2 \times D_\infty$.

4.5. **Four ideal vertices.** There are three such Coxeter groups, namely the groups $[3^{[\,]\times[\,]}]$, $[4^{[4]}]$ and $[(3,6)^{[2]}]$. It is clear that these groups have no periodic geodesics of type I, and hence the only maximal virtually infinite $\mathcal{VC}$ subgroups of hyperbolic type in both of these groups are isomorphic to $\mathbb{Z}$, D_∞, $\mathbb{Z}_2 \times \mathbb{Z}$, or $\mathbb{Z}_2 \times D_\infty$.

5. The algebraic K-theory of maximal finite subgroups.

The only (maximal) finite subgroups which occur inside the 32 groups we are interested in are, up to isomorphism, one of the following groups (see the previous section):

Maximal finite subgroups: 1, $\mathbb{Z}/2$, D_n for $n = 2, 3, 4, 5, 6, 10$ (here D_n denotes the dihedral group of order $2n$), $D_2 \times \mathbb{Z}/2$, $D_6 \times \mathbb{Z}/2$, S_4, $S_4 \times \mathbb{Z}/2$, and $A_5 \times \mathbb{Z}/2$.

Note that these groups are precisely the various finite parabolic subgroups (in the Coxeter sense) appearing amongst the 32 Coxeter groups we are considering. In the spectral sequence computing the homology $H_*^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$ the E^2-term is given by the algebraic K-groups of the finite subgroups. The non-trivial K-groups are listed in Table 5 at the end of this section.

For all but *four* of the finite groups in our list, their lower algebraic K-theory is well known. The relevant reference are listed below for each of these groups:

- $\mathbb{Z}/2$: For the negative K-groups, we refer the reader to [C80a]; the fact that $K_{-1}(\mathbb{Z}[\mathbb{Z}/2]) = 0$ can also be found in [Bas68, Theorem 10.6, page 695]. The vanishing of $\tilde{K}_0$ can be found in [CuR87, Corollary 5.17]. For information about the vanishing of Wh we refer the reader to [O89].
- $D_2 \times \mathbb{Z}/2$: For the negative K-groups, we refer the reader to [C80a]. The formula in Bass [Bas68, Chapter 12] shows also that $K_{-1}(\mathbb{Z}[D_2 \times \mathbb{Z}/2]) = 0$. The vanishing of $\tilde{K}_0$ can be found in [Re76], and for the information concerning Wh we refer the reader to [Ma78], [Ma80], [O89].
- $D_n,\ n = 2, 3, 4$: For the vanishing of the negative K-groups, we refer the reader to [C80a]. For the $\tilde{K}_0$ we refer the reader to [Re76]. In particular, the vanishing of $\tilde{K}_0(\mathbb{Z}G)$ is proven for $G = D_3$ in [Re76, Theorem 8.2] and for $G = D_4$ in [Re76, Theorem 6.4]. For information about Wh we refer the reader to [Ma78], [Ma80] and [O89].
- D_5: As far as we know the only K-groups found in the literature are $\tilde{K}_0(\mathbb{Z}D_5) \cong 0$ (see [Re76], [EM76]) and $K_q(\mathbb{Z}D_5) \cong 0$ for all $q \leq -2$ (see [C80a]). To compute $K_{-1}(\mathbb{Z}D_5)$, we used results that can be found in [C80a], and [C80b], and to compute $Wh(D_5)$, we used results that can be found in [Ma78], [Ma80] and [O89] (see the details in the Section 5.1).
- D_{10}: As far as we know the only K-groups found in the literature are $\tilde{K}_0(\mathbb{Z}D_{10}) \cong 0$ (see [Re76], [EM76]) and $K_q(\mathbb{Z}D_{10}) \cong 0$ for all $q \leq -2$ (see [C80a]). For the $K_{-1}(\mathbb{Z}D_{10})$, we used the results found in [C80a], and [C80b], and for $Wh(D_{10})$, we used results that can be found in [Ma78], [Ma80] and [O89] (see the details in the Section 5.2).
- D_6: See the discussion in Section 5.1. The whitehead groups $Wh_q(D_6)$ for $q \leq 1$ can also be found in [Pe98, Section 3], and [Or04, Section 5].
- $D_4 \times \mathbb{Z}/2$: Ortiz in [Or04, Section 5] using results from [C80a] [C80b], [CuR87], [O89] and [Ma06] showed that $K_q(\mathbb{Z}[D_4 \times \mathbb{Z}/2]) = 0,\ q \leq -1$, $\tilde{K}_0(\mathbb{Z}[D_4 \times \mathbb{Z}/2]) \cong \mathbb{Z}/4$, and that $Wh(D_4 \times \mathbb{Z}/2)$ is trivial.
- S_4: computed in [BFPP00]
- $S_4 \times \mathbb{Z}/2$: computed in [Or04, Section 5].

For the remaining groups in our list, we detail the computations in the next few subsections.

5.1. **The Lower algebraic K-theory of $D_6 \times \mathbb{Z}/2$.** Carter shows in [C80a] that for $q \leq -2$, $K_q(\mathbb{Z}F) = 0$ when F is a finite group. To calculate $K_{-1}(\mathbb{Z}[D_6 \times \mathbb{Z}/2])$, we use the following formula due to Carter [C80b, Theorem 3].

Let G be a group of order n, let p denote a prime number, let $\hat{\mathbb{Z}}_p$ denote the p-adic integers and let $\hat{\mathbb{Q}}_p$ denote the p-adic numbers. Then the following sequence is exact:

$$0 \to K_0(\mathbb{Z}) \to K_0(\mathbb{Q}G) \oplus \bigoplus_{p|n} K_0(\hat{\mathbb{Z}}_p G) \to \bigoplus_{p|n} K_0(\hat{\mathbb{Q}}_p G) \to K_{-1}(\mathbb{Z}G) \to 0.$$

The group algebra $\mathbb{Q}[D_6 \times \mathbb{Z}/2]$ is isomorphic to $\mathbb{Q}^8 \times (M_2(\mathbb{Q}))^4$ and the same statement is true if $\mathbb{Q}$ is replaced by $\hat{\mathbb{Q}}_2$ and $\hat{\mathbb{Q}}_3$. Hence $K_0(\mathbb{Q}[D_6 \times \mathbb{Z}/2]) \cong$

$K_0(\hat{\mathbb{Q}}_2[D_6 \times \mathbb{Z}/2]) \cong K_0(\hat{\mathbb{Q}}_3[D_6 \times \mathbb{Z}/2]) \cong \mathbb{Z}^{12}$. The integral p-adic terms are $K_0(\hat{\mathbb{Z}}_2[D_6 \times \mathbb{Z}/2]) \cong K_0(\mathbb{F}_2[D_6 \times \mathbb{Z}/2] \cong K_0(\mathbb{F}_2[D_6]) \cong \mathbb{Z}^2$, (see [Or04, page 350]), and $K_0(\hat{\mathbb{Z}}_3[D_6 \times \mathbb{Z}/2]) \cong K_0(\mathbb{F}_3[D_6 \times \mathbb{Z}/2]) \cong K_0(\mathbb{F}_3[(\mathbb{Z}/2)^3]) \cong \mathbb{Z}^8$. Carter also shows in [C80a] that $K_{-1}(\mathbb{Z}[D_6 \times \mathbb{Z}/2])$ is torsion free, so counting ranks in the exact sequence, we have that $K_{-1}(\mathbb{Z}[D_6 \times \mathbb{Z}/2]) \cong \mathbb{Z}^3$.

To compute $\tilde{K}_0(\mathbb{Z}[D_6 \times \mathbb{Z}/2])$, consider the following Cartesian square

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}/2][D_6] \arrow[r] \arrow[d] & \mathbb{Z}[D_6] \arrow[d] \\
\mathbb{Z}[D_6] \arrow[r] & \mathbb{F}_2[D_6]
\end{tikzcd}$$

which yields the Mayer-Vietories sequence (see [CuR87, Theorem 49.27])

(1)
$$K_1(\mathbb{Z}[D_6 \times \mathbb{Z}/2]) \to K_1(\mathbb{Z}D_6) \oplus K_1(\mathbb{Z}D_6) \xrightarrow{\varphi} K_1(\mathbb{F}_2[D_6]) \to$$
$$\to \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][D_6]) \to \tilde{K}_0(\mathbb{Z}D_6) \oplus \tilde{K}_0(\mathbb{Z}D_6) \to 0$$

We now proceed to compute the various terms appearing in this sequence.

We start by looking at the terms involving $\mathbb{Z}D_6$. In [Re76] Reiner shows that $\tilde{K}_0(\mathbb{Z}D_6)$ is trivial. $K_1(\mathbb{Z}D_6)$ can be computed as follows: since $Wh(G)$ equals $K_1(\mathbb{Z}G)/\{\pm G^{ab}\}$, the rank of $K_1(\mathbb{Z}G)$ is equal to the rank of $Wh(G)$. But the rank of $Wh(G)$ is $y = r - q$, where r denotes the number of irreducible real representations of G, and q denotes the number of irreducible rational representations of G. In [Bas65] Bass shows that r is equal to the number of conjugacy classes of sets $\{x, x^{-1}\}$, $x \in G$ and q is the number of conjugacy classes of cyclic subgroups of G (see also [Mi66]). For $G = D_6$, a direct calculations shows that $r = q$, and hence that $Wh(G)$ is purely torsion. Next note that the torsion part of $K_1(\mathbb{Z}G)$ is $\{\pm 1\} \oplus G^{ab} \oplus SK_1(\mathbb{Z}G)$ (see [W74]), and hence we have that $Wh(D_6) = SK_1(\mathbb{Z}D_6)$. Since Magurn [Ma78] has shown that $SK_1(\mathbb{Z}G)$ is trivial, we see that $Wh(D_6)$ is trivial. Since $(D_6)^{ab} = (\mathbb{Z}/2)^2$, we obtain that $K_1(\mathbb{Z}[D_6]) = (\mathbb{Z}/2)^3$.

Next we consider the remaining terms in the Mayer-Vietoris sequence. For $G = D_6 \times \mathbb{Z}/2$, Magurn in [Ma80, Corollary 11] shows that $Wh(D_6 \times \mathbb{Z}/2) = 0$ (note that the rank of $Wh(G \times \mathbb{Z}/2)$ is twice the rank of $Wh(G)$ since r and q get doubled, see Section 5.2 and 5.3). Since $(D_6 \times \mathbb{Z}/2)^{ab} = (\mathbb{Z}/2)^3$, this yields $K_1(\mathbb{Z}[D_6 \times \mathbb{Z}/2]) = (\mathbb{Z}/2)^4$. Finally, Magurn in [Ma06, Example 9]) shows that $K_1(\mathbb{F}_2[D_6]) = (\mathbb{Z}/2)^4$. Substituting all the known terms into the exact sequence in (1) yields the following exact sequence:

$$(2) \qquad (\mathbb{Z}/2)^4 \xrightarrow{\sigma} (\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/2)^3 \xrightarrow{\varphi} (\mathbb{Z}/2)^4 \to \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][D_6]) \to 0.$$

Next, we study the image of $\varphi : K_1(\mathbb{Z}D_6) \oplus K_1(\mathbb{Z}D_6) \to K_1(\mathbb{F}_2[D_6])$. We claim that $\operatorname{im}(\varphi) = (\mathbb{Z}/2)^2$. This can be seen as follows: first $\operatorname{im}(\varphi) = \operatorname{im}(\psi)$ where $\psi : K_1(\mathbb{Z}D_6) \longrightarrow K_1(\mathbb{F}_2[D_6])$ is induced by the canonical ring homomorphism $\mathbb{Z} \to \mathbb{F}_2$. Note the $K_1(\mathbb{Z})$ is a direct summand of $K_1(\mathbb{Z}D_6)$ and isomorphic to $\mathbb{Z}/2$; but this summand goes to zero in $K_1(\mathbb{F}_2[D_6])$ since it factors through the following

commutative square

$$\begin{array}{ccc}
\mathbb{Z}/2 = K_1(\mathbb{Z}) & \longrightarrow & K_1(\mathbb{F}_2) = 0 \\
\downarrow & & \downarrow \\
K_1(\mathbb{Z}D_6) & \longrightarrow & K_1(\mathbb{F}_2[D_6])
\end{array}$$

Since $K_1(\mathbb{Z}D_6) = (\mathbb{Z}/2)^3$, this forces $\dim_{\mathbb{F}_2}(\mathrm{im}(\varphi)) \leq 2$. Now from the exact sequences given in (1) and (2), we have that:

$$\dim_{\mathbb{F}_2}(\mathrm{im}(\varphi)) = 2\dim_{\mathbb{F}_2}(K_1(\mathbb{Z}[D_6])) - \dim_{\mathbb{F}_2}(\ker(\varphi)) = 6 - \dim_{\mathbb{F}_2}(\mathrm{im}(\sigma)).$$

Since $\dim_{\mathbb{F}_2}(\mathrm{im}(\sigma)) \leq 4$, we see that $\dim_{\mathbb{F}_2}(\mathrm{im}(\varphi)) \geq 2$, which forces $\mathrm{im}(\varphi) \cong (\mathbb{Z}/2)^2$. The exact sequence now yields $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][D_6]) \cong (\mathbb{Z}/2)^2$.

5.2. **The computation of the K-groups $K_{-1}(\mathbb{Z}D_5)$, and $Wh(D_5)$.** To compute $K_{-1}(\mathbb{Z}D_5)$, we need Carter's formula for K_{-1}, [C80b, Theorem 3], the reader is referred to Section 5.1.

$$0 \to K_0(\mathbb{Z}) \to K_0(\mathbb{Q}D_5) \oplus \bigoplus_{p|n} K_0(\hat{\mathbb{Z}}_p D_5) \to \bigoplus_{p|n} K_0(\hat{\mathbb{Q}}_p D_5) \to K_{-1}(\mathbb{Z}D_5) \to 0.$$

The group algebra $\mathbb{Q}D_5$ is isomorphic to $\mathbb{Q} \times \mathbb{Q}$, and the same statement is true if $\mathbb{Q}$ is replaced by $\hat{\mathbb{Q}}_2$ (recall that $\sqrt{5} \notin \hat{\mathbb{Q}}_2$). For $p = 5$, the group algebra $\hat{\mathbb{Q}}_5 D_5 \cong (\hat{\mathbb{Q}}_5)^2 \times M_2(\hat{\mathbb{Q}}_5)$. Hence $K_0(\hat{\mathbb{Q}}_2[D_5]) \cong K_0(\mathbb{Q}[D_5]) \cong \mathbb{Z}^2$, and $K_0(\hat{\mathbb{Q}}_5[D_5]) \cong \mathbb{Z}^3$. Using techniques described in [CuR81, Section 5], we have that $K_0(\hat{\mathbb{Z}}_5[D_5]) \cong K_0(\mathbb{F}_5[D_5]) \cong K_0(\mathbb{F}_5[\mathbb{Z}/2]) = K_0(\mathbb{F}_5 \times \mathbb{F}_5) = \mathbb{Z}^2$. Also $K_0(\hat{\mathbb{Z}}_2[D_5]) \cong K_0(\mathbb{F}_2[D_5]) \cong K_0(\mathbb{F}_2 \times M_2(\mathbb{F}_2)) = \mathbb{Z}^2$. Carter also shows in [C80a] that $K_{-1}(\mathbb{Z}[D_5])$ is torsion free, so counting ranks as before, we have that $K_{-1}(\mathbb{Z}D_5) \cong 0$.

Next, we compute $Wh(D_5)$. Recall that $Wh(G) = \mathbb{Z}^y \oplus SK_1(\mathbb{Z}G)$. Magurn in [Ma78] proves that SK_1 vanishes for all finite dihedral groups. The rank of the torsion free part is $y = r - q$ (see Section 5.1). Since in $D_5 = \langle r, s \,|\, r^5 = s^2 = 1, srs = r^{-1}\rangle$, there are three cyclic subgroups modulo conjugacy (the trivial subgroup $\{e\}$, C_2 and C_5) and four conjugacy classes of sets $\{x, x^{-1}\}$ (consisting of $\{e\}$, $\{r, r^4\}$, $\{r^2, r^3\}$ and $\{s, s\}$), we see that $r = 4$ and $q = 3$. This yields $Wh(D_5) \cong \mathbb{Z}^{r-q} \cong \mathbb{Z}$.

5.3. **The computation of the K-groups $K_{-1}(\mathbb{Z}D_{10})$, and $Wh(D_{10})$.** Carter shows in [C80a] that for $q \leq -2$, $K_q(\mathbb{Z}D_{10}) = 0$. To calculate $K_{-1}(\mathbb{Z}D_{10})$, again using Carter's formula for K_{-1} [C80b, Theorem 3], we have (see Section 5.1 and 5.2):

$$0 \to K_0(\mathbb{Z}) \to K_0(\mathbb{Q}D_{10}) \oplus \bigoplus_{p|n} K_0(\hat{\mathbb{Z}}_p D_{10}) \to \bigoplus_{p|n} K_0(\hat{\mathbb{Q}}_p D_{10}) \to K_{-1}(\mathbb{Z}D_{10}) \to 0$$

The group algebra $\mathbb{Q}[D_5 \times \mathbb{Z}/2]$ is isomorphic to $\mathbb{Q}^4$ and the same statement is true if $\mathbb{Q}$ is replaced by $\hat{\mathbb{Q}}_2$. For $p = 5$, the group algebra $\hat{\mathbb{Q}}_5[D_5 \times \mathbb{Z}/2] \cong (\hat{\mathbb{Q}}_5)^4 \times (M_2(\hat{\mathbb{Q}}_5))^2$. Hence $K_0(\mathbb{Q}[D_5 \times \mathbb{Z}/2]) \cong K_0(\hat{\mathbb{Q}}_2[D_5 \times \mathbb{Z}/2]) \cong \mathbb{Z}^4$, and $K_0(\hat{\mathbb{Q}}_5[D_5 \times \mathbb{Z}/2]) \cong \mathbb{Z}^6$. The integral p-adic terms are $K_0(\hat{\mathbb{Z}}_2[D_5 \times \mathbb{Z}/2]) \cong K_0(\mathbb{F}_2[D_5 \times \mathbb{Z}/2]) \cong K_0(\mathbb{F}_2[D_5]) \cong \mathbb{Z}^2$ (see Section 5.2), and $K_0(\hat{\mathbb{Z}}_5[D_5 \times \mathbb{Z}/2]) \cong K_0(\mathbb{F}_5[D_5 \times \mathbb{Z}/2]) \cong K_0(\mathbb{F}_5[(\mathbb{Z}/2)^2]) \cong \mathbb{Z}^4$. Carter also shows in [C80a] that $K_{-1}(\mathbb{Z}[D_{10}])$ is torsion free, so counting ranks in the exact sequence, we have that $K_{-1}(\mathbb{Z}D_{10}) \cong \mathbb{Z}$.

Next, we compute $Wh(D_{10})$. Magurn in [Ma78] proves that SK_1 vanishes for all finite dihedral groups. For $D_{10} = \langle r, s \,|\, r^{10} = s^2 = 1, srs = r^{-1}\rangle$, we have four cyclic subgroups (modulo conjugacy) of D_{10} : the trivial subgroup $\{e\}$, $\langle s\rangle$, $\langle r^2\rangle$ and $\langle r\rangle$. On the other hand there are six conjugacy classes of sets $\{x, x^{-1}\}$: $\{e\}$, $\{r, r^9\}$, $\{r^2, r^8\}$, $\{r^3, r^7\}$, $\{r^4, r^6\}$, $\{r^5, r^5\}$ and $\{s, s\}$. This gives us $r = 6$, $q = 4$, and hence $Wh(D_{10}) \cong \mathbb{Z}^{r-q} \cong \mathbb{Z}^2$.

5.4. **The Lower algebraic K-theory of $A_5 \times \mathbb{Z}/2$.** Carter shows in [C80a] that for $q \le -2$, $K_q(\mathbb{Z}F) = 0$ when F is a finite group. To compute $Wh_q(A_5 \times \mathbb{Z}/2)$ for $q \le 1$, we first claim that

$$Wh_q(A_5) = \begin{cases} \mathbb{Z} & q = 1 \\ 0 & q = 0 \\ 0 & q \le -1. \end{cases}$$

This can be seen as follows: by [O89, Theorem 14.6], we have that $SK_1(\mathbb{Z}A_5) = 0$. The group A_5 has precisely five (mutually nonisomorphic) irreducible real representation, giving $r = 5$. In A_5 the conjugacy classes of cyclic subgroups are represented by the trivial subgroup $\{e\}$, $\langle (12)(34)\rangle$, $\langle (123)\rangle$, $\langle (12345)\rangle$ giving us that $q = 4$. This forces $Wh(A_5) \cong \mathbb{Z}^{r-q} = \mathbb{Z}$. By [EM76], we have that $\tilde{K}_0(\mathbb{Z}A_5) = 0$; Dress induction as used in [O89, Theorem 11.2] shows that $K_{-1}(\mathbb{Z}A_5) = 0$, and by [C80a] we have that $K_q(\mathbb{Z}A_5) = 0$ for $q \le -2$.

Now let us compute $Wh(A_5 \times \mathbb{Z}/2)$. Magurn in [Ma, Example 5] shows that $SK_1(\mathbb{Z}[A_5 \times \mathbb{Z}/2]) = 0$. Since $\operatorname{rank}(Wh(G \times \mathbb{Z}/2)) = 2\operatorname{rank}(Wh(G))$ and $Wh(A_5) \cong \mathbb{Z}$ we get $Wh(A_5 \times \mathbb{Z}/2) \cong \mathbb{Z}^2$.

Next, we compute $K_{-1}(\mathbb{Z}[A_5 \times \mathbb{Z}/2])$. Consider the following Cartesian square

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}/2][A_5] & \longrightarrow & \mathbb{Z}A_5 \\ \downarrow & & \downarrow \\ \mathbb{Z}A_5 & \longrightarrow & \mathbb{F}_2[A_5] \end{array}$$

which yields the Mayer-Vietories sequence (see [CuR87, Theorem 49.27])

$$\begin{aligned} \ldots \to \tilde{K}_0(\mathbb{Z}A_5) \oplus \tilde{K}_0(\mathbb{Z}A_5) \xrightarrow{\varphi} \tilde{K}_0(\mathbb{F}_2[A_5]) \to \\ \to K_{-1}(\mathbb{Z}[\mathbb{Z}/2][A_5]) \to K_{-1}(\mathbb{Z}A_5) \oplus K_{-1}(\mathbb{Z}A_5) \to \cdots \end{aligned} \tag{3}$$

from which we first obtain $K_{-1}(\mathbb{Z}[A_5 \times \mathbb{Z}/2]) \cong \tilde{K}_0(\mathbb{F}_2[A_5])$. Since $\mathbb{F}_2[A_5] \cong \mathbb{F}_2 \times M_4(\mathbb{F}_2)$, we have that $K_0(\mathbb{F}_2[A_5]) \cong \mathbb{Z}^2$, from which it follows that $K_{-1}(\mathbb{Z}[A_5 \times \mathbb{Z}/2]) \cong \mathbb{Z}$.

Next, we claim that $\tilde{K}_0(\mathbb{Z}[A_5 \times \mathbb{Z}/2])$ is trivial. To see this, let H be a subgroup of G. For any locally free $\mathbb{Z}G$-module M its restriction to H (denoted by M_H) is a locally free $\mathbb{Z}H$-module. The mapping defined by $[M] \to [M_H]$ gives a homomorphism of $\tilde{K}_0(\mathbb{Z}G) \to \tilde{K}_0(\mathbb{Z}H)$.

A group H is *hyper-elementary* if H is a semidirect product $N \rtimes P$ of a cyclic normal subgroup N and a subgroup P of prime order, where $(|N|, |P|) = 1$. Denote $\mathcal{H}(G)$ the full set of non-conjugate hyper-elementary subgroups of G. We shall need

the following result due to Swan (see [Sw60]): the map

$$\tilde{K}_0(\mathbb{Z}G) \longrightarrow \prod_{H\in\mathcal{H}(G)} \tilde{K}_0(\mathbb{Z}H) \tag{4}$$

is a monomorphism for any finite group G. In particular, if $\tilde{K}_0(\mathbb{Z}H) = 0$ for all $H \in \mathcal{H}$, then we immediately obtain $\tilde{K}_0(\mathbb{Z}G) = 0$.

To begin the proof, we first list a full set $\mathcal{H}(A_5)$ of non-conjugate hyper-elementary subgroups of A_5: $\mathbb{Z}/2\times\mathbb{Z}/2$, D_3 and D_5. Note that the hyper-elementary subgroups of $G \times \mathbb{Z}/2$ are of the form H or $H \times \mathbb{Z}/2$ for $H \in \mathcal{H}(G)$. In particular, the non-conjugate hyper-elementary subgroups of $A_5 \times \mathbb{Z}/2$ are: $\mathbb{Z}/2 \times \mathbb{Z}/2$, D_3 and D_5, $(\mathbb{Z}/2)^3$, $D_3 \times \mathbb{Z}/2 \cong D_6$, and $D_5 \times \mathbb{Z}/2 \cong D_{10}$.

By the results already mentioned in Sections 5.1, 5.2, 5.3, we have $\tilde{K}_0(\mathbb{Z}H) = 0$ for all $H \in \mathcal{H}(A_5 \times \mathbb{Z}/2)$. The result of Swan on the injectivity of the map in (4) immediately implies that $\tilde{K}_0(\mathbb{Z}[A_5 \times \mathbb{Z}/2]) = 0$.

$Q \in \mathcal{VC}$	$Wh_q \neq 0,\ q \leq -1$	$\tilde{K}_0 \neq 0$	$Wh \neq 0$
D_5			$\mathbb{Z}$
D_6	$K_{-1} \cong \mathbb{Z}$		
$D_4 \times \mathbb{Z}/2$		$\mathbb{Z}/4$	
D_{10}	$K_{-1} \cong \mathbb{Z}$		$\mathbb{Z}^2$
$D_6 \times \mathbb{Z}/2$	$K_{-1} \cong \mathbb{Z}^3$	$(\mathbb{Z}/2)^2$	
$S_4 \times \mathbb{Z}/2$	$K_{-1} \cong \mathbb{Z}$	$\mathbb{Z}/4$	
A_5			$\mathbb{Z}$
$A_5 \times \mathbb{Z}/2$	$K_{-1} \cong \mathbb{Z}$		$\mathbb{Z}^2$

Table 5: Lower algebraic K-theory of subgroups $Q \in \mathcal{FIN}$

6. Cokernels of relative assembly maps for maximal infinite virtually cyclic subgroups

In view of Corollary 3.4, we will need for our computations the cokernels of the relative assembly maps for the various maximal infinite virtually cyclic subgroups of Type I. From the tables 2, 3, 4 computed in Section 4, we have the following list containing *all* the maximal infinite virtually cyclic subgroups that appear in the 32 groups we are interested in:

Maximal infinite virtually cyclic subgroups: $\mathbb{Z}$, D_∞, $\mathbb{Z} \times \mathbb{Z}/2$, $D_\infty \times \mathbb{Z}/2$, $D_n \times D_\infty$, for $n = 2, 3, 4, 5$, $D_4 *_{D_2} D_4$, and $D_2 \times \mathbb{Z}/2 *_{D_2} D_4$.

We first note that, for the groups in our list, the cokernels are known to be trivial in the following cases:

- $\mathbb{Z}$: by work of Bass [Bas68].

- D_∞: by work of Waldhausen [Wd78].
- $\mathbb{Z} \times \mathbb{Z}/2$, and $D_\infty \times \mathbb{Z}/2$: by work of Pearson [Pe98, Section 2].
- $D_3 \times D_\infty$: by work of the authors [LO, Section 4]

Finally, the authors have also shown in [LO, Section 4] that for the group $D_2 \times D_\infty$, the cokernels of the relative assembly map for $n = 0, 1$ are countably infinite direct sums of $\mathbb{Z}/2$. The remaining four groups in our list will be discussed in the following subsections.

Observe that by a result of Farrell and Jones [FJ95], the cokernels we are interested in $H_n^V(E_{\mathcal{FIN}}(V) \to *)$ are automatically trivial for $n \leq -1$. In particular, we only need to focus on the cases $n = 0$, and $n = 1$. These cokernels are precisely the elusive Bass, Farrell, and Waldhausen Nil-groups. We are able to identify these cokernels exactly, with the exception of the case $D_4 \times D_\infty$. For this group, we content ourselves with summarizing what we were able to obtain in Subsection 6.4. We summarize the non-trivial cokernels in Table 6.

$V \in \mathcal{VC}$	$H_0^V(E_{\mathcal{FIN}}(V) \to *) \neq 0$	$H_1^V(E_{\mathcal{FIN}}(V) \to *) \neq 0$
$D_2 \times D_\infty$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$D_4 *_{D_2} D_4$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(D_2 \times \mathbb{Z}/2) *_{D_2} D_4$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$D_4 \times D_\infty$	Nil_0	Nil_1

Table 6: Cokernels of relative assembly map for maximal $V \in \mathcal{VC}$

6.1. **The Lower algebraic K-theory of $D_5 \times D_\infty$.** First, note that $D_5 \times D_\infty \cong D_{10} *_{D_5} D_{10}$. As before $K_n(\mathbb{Z}Q)$ is zero for $n < -1$ (see [FJ95]). Since $K_{-1}(\mathbb{Z}D_5) = 0$ (see Section 5.2), and $K_{-1}(\mathbb{Z}D_{10}) = \mathbb{Z}$ (see Section 5.3), we see that for $Q = D_{10} *_{D_5} D_{10}$, we have $K_{-1}(\mathbb{Z}Q) = \mathbb{Z} \oplus \mathbb{Z}$.

For the remaining K-groups, we make use of [CP02, Lemma 3.8]. For $Q = D_{10} *_{D_5} D_{10}$, $\tilde{K}_0(\mathbb{Z}Q) \cong NK_0(\mathbb{Z}D_5; C_1, C_2)$, where $C_i = \mathbb{Z}[D_{10} - D_5]$ is the $\mathbb{Z}D_5$-bimodule generated by $D_{10} - D_5$ for $i = 1, 2$, (see Section 5.2 and 5.3 for the $\tilde{K}_0(\mathbb{Z}D_n)$ for $n = 5, 10$), and $Wh(Q) \cong \mathbb{Z}^3 \oplus NK_1(\mathbb{Z}D_5; C_1, C_2)$, with C_1 and C_2 as before, (see Section 5.2 and 5.3 for the $Wh(D_n)$ for $n = 5, 10$). The Nil-groups appearing in these computations are the Waldhausen's Nil-groups.

Now by [LO(b)], we know that $NK_i(\mathbb{Z}D_5; C_1, C_2) = 0$ for $i = 0, 1$, vanishes if and only if the corresponding Farrell Nil-group vanishes for the canonical index two subgroup $D_5 \times \mathbb{Z} \triangleleft D_5 \times D_\infty$. Note that in this case, the Farrell Nil-group is untwisted, and hence is just the Bass Nil-group $NK_i(\mathbb{Z}D_5)$. But Harmon [Ha87] has shown that for finite groups G of square-free order (such as D_5), the Bass Nil group $NK_i(\mathbb{Z}G)$ vanishes for $i = 0, 1$. We summarize our computations in the

following:

$$Wh_q(D_5 \times D_\infty) = \begin{cases} \mathbb{Z}^3 & q = 1 \\ 0 & q = 0 \\ \mathbb{Z}^2 & q = -1 \\ 0 & q \leq -2. \end{cases}$$

6.2. **The Lower algebraic K-theory of $D_2 \times \mathbb{Z}/2 *_{D_2} D_4$.** As before $K_n(\mathbb{Z}Q)$ is zero for $n < -1$ (see [FJ95]). Since $K_{-1}(\mathbb{Z}D_2) = 0$, $K_{-1}(\mathbb{Z}[D_2 \times \mathbb{Z}/2]) = 0$, and $K_{-1}(\mathbb{Z}D_4) = 0$, we see that for $Q = D_2 \times \mathbb{Z}/2 *_{D_2} D_4$, we have that $K_{-1}(\mathbb{Z}Q) = 0$.

For the remaining K-groups using [CP02, Lemma 3.8], we have that for $Q = D_2 \times \mathbb{Z}/2 *_{D_2} D_4$, $\tilde{K}_0(\mathbb{Z}Q) \cong NK_0(\mathbb{Z}D_2; A_1, A_2)$, where $A_1 = \mathbb{Z}[D_2 \times \mathbb{Z}/2 - D_2]$ is the $\mathbb{Z}D_2$ bi-module generated by $(D_2 \times \mathbb{Z}/2) - D_2$, and $A_2 = \mathbb{Z}[D_4 - D_2]$ is the $\mathbb{Z}D_2$ bi-module generated by $D_4 - D_2$. Similarly, we have that $Wh(Q) \cong NK_1(\mathbb{Z}D_2; A_1, A_2)$, where A_1, A_2 are the bi-modules defined above.

Now recall that in [LO, Theorem 5.2], the authors established that (1) $\tilde{K}_0(\mathbb{Z}[D_2 \times D_\infty]) \cong \bigoplus_\infty \mathbb{Z}/2$ and (2) $K_1(\mathbb{Z}[D_2 \times D_\infty]) \cong \bigoplus_\infty \mathbb{Z}/2$. The computation reduced to showing that the Waldhausen Nil-groups $NK_i(\mathbb{Z}D_2; A_2, A_2)$ is isomorphic to an infinite countable sum of $\mathbb{Z}/2$ (where the bi-module A_2 is defined in the previous paragraph). This was achieved by establishing (1) the existence of an injection, and (2) the existence of a (different) surjection, from the Bass Nil-group $NK_i(\mathbb{D}_4) \cong \bigoplus_\infty \mathbb{Z}/2$ into the corresponding Waldhausen Nil-group $NK_i(\mathbb{Z}D_4; A_2, A_2)$. But the reader can verify that the argument given in [LO] applies verbatim to the Waldhausen Nil-groups $NK_i(\mathbb{Z}D_4; A_1, A_2)$ appearing in our present computation.

We conclude that the lower algebraic K-theory of $D_2 \times \mathbb{Z}/2 *_{D_2} D_4$ is given by:

$$Wh_q(D_2 \times \mathbb{Z}/2 *_{D_2} D_4) = \begin{cases} \bigoplus_\infty \mathbb{Z}/2 & q = 1 \\ \bigoplus_\infty \mathbb{Z}/2 & q = 0 \\ 0 & q \leq -1. \end{cases}$$

6.3. **The Lower algebraic K-theory of $D_4 *_{D_2} D_4$.** As before $K_n(\mathbb{Z}Q)$ is zero for $n < -1$ (see [FJ95]). Since $K_{-1}(\mathbb{Z}D_2) = 0$ and $K_{-1}(\mathbb{Z}D_4) = 0$, we see that for $Q = D_4 *_{D_2} D_4$, we have that $K_{-1}(\mathbb{Z}Q) = 0$.

For the remaining K-groups, using [CP02, Lemma 3.8], we have that for $Q = D_4 *_{D_2} D_4$, $\tilde{K}_0(\mathbb{Z}Q) \cong NK_0(\mathbb{Z}D_2; F_1, F_2)$, where for $i = 1, 2$, $F_i = \mathbb{Z}[D_4 - D_2]$ is the $\mathbb{Z}D_2$ bi-module generated by $D_4 - D_2$. Similarly, we have that $Wh(Q) \cong NK_1(\mathbb{Z}D_2; F_1, F_2)$, with F_1 and F_2 as before.

Now using [LO, Theorem 5.2], we concluded that for $Q = D_4 *_{D_2} D_4$

$$Wh_q(Q) = \begin{cases} \bigoplus_\infty \mathbb{Z}/2 & q = 1 \\ \bigoplus_\infty \mathbb{Z}/2 & q = 0 \\ 0 & q \leq -1. \end{cases}$$

6.4. **The Lower algebraic K-theory of $D_4 \times D_\infty$.** The authors were unable to obtain an explicit computation for this group. In this case, we have that $D_4 \times D_\infty \cong (D_4 \times \mathbb{Z}/2) *_{D_4} (D_4 \times \mathbb{Z}/2)$, and we are interested in the Waldhausen Nil-groups associated to this splitting. A special case of recent independent work of several authors (including H. Reich, F. Quinn, J. Davis and A. Ranicki) is that

this Waldhausen Nil-group is isomorphic to the Bass Nil-group associated to the canonical index two subgroup $D_4 \times \mathbb{Z}$ inside $D_4 \times D_\infty$ (a considerable strengthening of the result of the authors in [LO(b)]). In our tables, we denote these groups by $Nil_0 = NK_0(\mathbb{Z} D_4)$ and $Nil_1 = NK_1(\mathbb{Z} D_4)$ (the lower Nil groups vanish by work of Farrell-Jones [FJ95]). These abelian groups are known to have the following properties:

(1) Nil_1 is either trivial or infinitely generated [F77],
(2) Nil_0 is infinitely generated (see below),
(3) in both of these groups, the order of every element divides 8 ([CP02], [G07]).

It is very likely that the group Nil_1 is also non-trivial, but we were unable to establish this result. In order to see that Nil_0 is non-trivial, consider the following Cartesian square:

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}/4] \cong \mathbb{Z}[a]/a^4 - 1 = 0 & \longrightarrow & \mathbb{Z}[a]/a^2 - 1 = 0 \cong \mathbb{Z}[\mathbb{Z}/2] \\ \downarrow & & \downarrow \\ \mathbb{Z}[i] \cong \mathbb{Z}[a]/a^2 + 1 = 0 & \longrightarrow & \mathbb{F}_2[a]/a^2 - 1 = 0 \cong \mathbb{F}_2[\mathbb{Z}/2] \end{array}$$

which yields the Cartesian square for $\mathbb{Z}[D_4] = \mathbb{Z}[\mathbb{Z}/4 \rtimes_\alpha \mathbb{Z}/2] = \mathbb{Z}[\mathbb{Z}/4]_\alpha[\mathbb{Z}/2]$:

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}/4]_\alpha[\mathbb{Z}/2] & \longrightarrow & \mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/2] \\ \downarrow & & \downarrow \\ \mathbb{Z}[i]_\alpha[\mathbb{Z}/2] & \longrightarrow & \mathbb{F}_2[\mathbb{Z}/2][\mathbb{Z}/2] \end{array}$$

where in $\mathbb{Z}[i]_\alpha[\mathbb{Z}/2]$, the automorphism α acts via $\alpha(i) = -i$. Writing $D_2 = \mathbb{Z}/2 \times \mathbb{Z}/2$ and $A = \mathbb{Z}[i]_\alpha[\mathbb{Z}/2]$, and applying the NK-functor, this Cartesian square yields the Mayer-Vietoris sequence:

$$\begin{aligned} NK_2(\mathbb{F}_2[D_2]) \to & Nil_1 \to NK_1(\mathbb{Z}[D_2]) \oplus NK_1(A) \to NK_1(\mathbb{F}_2[D_2]) \to \\ & \to Nil_0 \to NK_0(\mathbb{Z}[D_2]) \oplus NK_0(A) \to NK_0(\mathbb{F}_2[D_2]) \end{aligned}$$

Several of the groups appearing in this Mayer-Vietoris sequence are known: the group $NK_0(\mathbb{F}_2[D_2])$ vanishes by [Bas68], while the authors have previously shown [LO] that the groups $NK_1(\mathbb{Z}[D_2])$ and $NK_0(\mathbb{Z}[D_2])$ are likewise countable infinite sums of $\mathbb{Z}/2$.

Focusing on the tail end of the Mayer-Vietoris sequence, and substituting in the expressions we already know, we see that:

$$\ldots \to Nil_0 \to NK_0(A) \oplus \bigoplus_\infty \mathbb{Z}/2 \to 0$$

and non-triviality of Nil_0 follows from the surjectivity onto the countable infinite sum of $\mathbb{Z}/2$. In contrast, focusing on the head of the Mayer-Vietoris sequence, we see that:

$$NK_2(\mathbb{F}_2[D_2]) \to Nil_1 \to NK_1(A) \oplus \bigoplus_\infty \mathbb{Z}/2 \to NK_1(\mathbb{F}_2[D_2]) \to \ldots$$

Hence to establish that Nil_1 is non-trivial from this sequence, one would need to either:

- establish that the first map is non-zero, i.e. understand the map

$$NK_2(\mathbb{F}_2[D_2]) \to NK_1(\mathbb{Z}\, D_4)$$

- establish that the second map is non-zero by showing that the third map has a non-trivial kernel, for instance by understanding the map $NK_1(\mathbb{Z}[D_2]) \to NK_1(\mathbb{F}_2[D_2])$

The authors have some partial results concerning some of the terms showing up in the head of the Mayer-Vietoris sequence, but so far have been unsuccessful in establishing non-triviality of Nil_1.

7. The spectral sequences and final computations

We now proceed to apply Corollary 3.4 to compute the lower algebraic K-theory of $\mathbb{Z}\,\Gamma$, for Γ one of the 32 possible 3-simplex hyperbolic reflection groups. Let us recall that Corollary 3.4 tells us that for such groups Γ, we have for $n \leq 1$ an isomorphism:

$$K_n(\mathbb{Z}\,\Gamma) \cong H_n^{\Gamma}(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{i=1}^{k} H_n^{V_i}(E_{\mathcal{FIN}}(V_i) \to *)$$

where $\{V_i\}_{i=1}^k$ are a complete set of representatives for the conjugacy classes of maximal infinite virtually cyclic subgroups of Type I.

We first note that for all 32 of our groups, we have:

- obtained in Section 4 a complete list of the Type I maximal infinite virtually cyclic subgroups (listed out in Tables 2, 3, and 4).
- computed in Section 6 the groups

$$H_n^V(E_{\mathcal{FIN}}(V) \to *)$$

  for all the Type I maximal infinite virtually cyclic subgroups that occur, with the exception of the case $V = D_4 \times D_\infty$.

In particular, this allows us to determine the expression

$$\bigoplus_{i=1}^{k} H_n^{V_i}(E_{\mathcal{FIN}}(V_i) \to *)$$

occurring in the formula above for all 32 of our groups.

Hence we are left with computing $H_n^{\Gamma}(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty})$ for each of our 32 groups. In order to do this, we recall that Quinn [Qu82] established the existence of a spectral sequence which converges to this homology group, with E^2-terms given by:

$$E^2_{p,q} = H_p(E_{\mathcal{FIN}}(\Gamma)/\,\Gamma\,; \{Wh_q(\Gamma_\sigma)\}) \Longrightarrow Wh_{p+q}(\Gamma).$$

The complex that gives the homology of $E_{\mathcal{VC}}(\Gamma)/\,\Gamma$ with local coefficients $\{Wh_q(\Gamma_\sigma)\}$ has the form

$$\cdots \to \bigoplus_{\sigma^{p+1}} Wh_q(\Gamma_{\sigma^{p+1}}) \to \bigoplus_{\sigma^p} Wh_q(\Gamma_{\sigma^p}) \to \bigoplus_{\sigma^{p-1}} Wh_q(\Gamma_{\sigma^{p-1}}) \cdots \to \bigoplus_{\sigma^0} Wh_q(\Gamma_{\sigma^0}),$$

where σ^p denotes the cells in dimension p, and the sum is over all p-dimensional cells in $E_{\mathcal{VC}}(\Gamma)/\,\Gamma$. The p^{th} homology group of this complex will give us the entries

for the $E^2_{p,q}$-term of the spectral sequence. Let us recall that

$$Wh_q(F) = \begin{cases} Wh(F), & q = 1 \\ \tilde{K}_0(\mathbb{Z}F), & q = 0 \\ K_q(\mathbb{Z}F), & q \leq -1. \end{cases}$$

Observe that for the groups we are interested it is particularly easy to obtain a model for $E_{\mathcal{FIN}}(\Gamma)$ with the additional nice property that the Γ-action is cocompact. Indeed, in the case of the 9 cocompact lattices, one can just take the Γ-space to be $\mathbb{H}^3$, with fundamental domain a 3-dimensional simplex. In the case of the 23 non-uniform lattices, one can Γ-equivariantly remove a disjoint collection of horoballs from $\mathbb{H}^3$ to form a Γ-space X_Γ on which Γ acts cocompactly. A fundamental domain for the Γ-action on the space X_Γ can be obtained by (1) taking the 3-simplex fundamental domain Δ^3_Γ for the Γ-action on $\mathbb{H}^3$, and (2) removing a small neighborhood of each ideal vertex in Δ^3_Γ.

For the resulting fundamental domain X_Γ/Γ, it is particularly easy to identify the stabilizers of each cell. Indeed, there will always be a single 3-dimensional cell, with trivial isotropy. The 2-dimensional cells will consist of

(1) precisely four cells corresponding to the original faces of Δ^3_Γ, each of which will have stabilizer $\mathbb{Z}/2$,
(2) one additional cell for each ideal vertex (obtained from "truncating" the vertex), with trivial stabilizer.

Note that since $Wh_q(1)$ and $Wh_q(\mathbb{Z}/2)$ vanish for all $q \leq 1$, this in particular implies that *there will never be any contribution to the E^2-terms from the 3-dimensional and 2-dimensional cells.* In other words, $E^2_{p,q} = 0$ except possibly for $p = 0, 1$.

Now let us focus on the 1-dimensional and 0-dimensional cells in the fundamental domain X_Γ. The 1-dimensional cells will consist of

(1) precisely six edges, corresponding to the original edges of Δ^3_Γ, each of which will have stabilizer a dihedral group D_n ($n = 2, 3, 4, 5$, or 6).
(2) three new edges for each ideal vertex (obtained from "truncating the vertex), with stabilizer $\mathbb{Z}/2$.

Note that amongst these groups, the only ones that have some non-trivial Wh_q are the groups D_5 (for $q = 1$) and D_6 (for $q = -1$). Now the 0-dimensional cells that occur consist of

(1) one vertex for each of the non-ideal vertices in Δ^3_Γ. Each of these vertices will have stabilizer a spherical Coxeter group, isomorphic to the special subgroup of the Coxeter group Γ which corresponds to the vertex (up to isomorphism, these are the groups occuring in Table 1).
(2) one new vertex for each edge leading into an ideal vertex (obtained from "truncating the cusp"). The stabilizer of the vertex will coincide with the stabilizer of the corresponding edge (and hence be a dihedral group D_n where $n = 2, 3, 4, 5$, or 6)

For all these groups, the non-vanishing Wh_q can be found in Table 5.

Finally, we observe that since the only 1-cells with non-trivial Wh_q are the groups D_5 and D_6, most of the morphisms in the chain complex for the E^2-terms will either be zero (or in a few cases, will clearly be isomorphisms). The three morphisms one needs to take care with are:

- $K_{-1}(\mathbb{Z}\, D_6) \to K_{-1}(\mathbb{Z}[D_6 \times \mathbb{Z}/2])$,
- $Wh(D_5) \to Wh(D_{10})$,
- $Wh(D_5) \to Wh(A_5 \times \mathbb{Z}/2)$.

We proceed to analyze each of these three morphisms in the next three sections.

7.1. **The map $K_{-1}(\mathbb{Z}\, D_6) \to K_{-1}(\mathbb{Z}[D_6 \times \mathbb{Z}/2])$.** We start by observing that $K_{-1}(\mathbb{Z}\, D_6) \cong \mathbb{Z}$ and $K_{-1}(\mathbb{Z}[D_6 \times \mathbb{Z}/2]) \cong \mathbb{Z}^3$ (see Table 5). We claim that the map induced by the natural inclusion $D_6 \hookrightarrow D_6 \times \mathbb{Z}/2$ is injective, and the quotient group is isomorphic to $\mathbb{Z}^2$. In order to see this, we merely note that there is a retraction from $D_6 \times \mathbb{Z}/2$ to the subgroup D_6, and hence we must have that $K_{-1}(\mathbb{Z}\, D_6) \cong \mathbb{Z}$ is a summand inside $K_{-1}(\mathbb{Z}[D_6 \times \mathbb{Z}/2]) \cong \mathbb{Z}^3$, which immediately gives our claim.

7.2. **The map $Wh(D_5) \to Wh(D_{10})$.** We start by observing that $Wh(D_5) \cong \mathbb{Z}$ and $Wh(D_{10}) \cong \mathbb{Z}^2$ (see Table 5). We claim that the map induced by the natural inclusion $D_5 \hookrightarrow D_{10} \cong D_5 \times \mathbb{Z}/2$ is injective, and the quotient group is isomorphic to $\mathbb{Z}$. But again, we see that there is a retraction from $D_5 \times \mathbb{Z}/2$ to the subgroup D_5, and hence $Wh(D_5) \cong \mathbb{Z}$ is a summand inside $Wh(D_{10}) \cong \mathbb{Z}^2$, which gives us our claim. Note that this map was used implicitly in Section 6.1 (in the argument mentioned in the second paragraph).

7.3. **The map $Wh(D_5) \to Wh(A_5 \times \mathbb{Z}/2)$.** We start by observing that $Wh(D_5) \cong \mathbb{Z}$ and $Wh(A_5 \times \mathbb{Z}/2) \cong \mathbb{Z}^2$ (see Table 5). We claim that the map induced by the natural inclusion $D_5 \hookrightarrow A_5 \times \mathbb{Z}/2$ is injective, and the quotient group is isomorphic to $\mathbb{Z}$. Note that in this case we do *not* have a retraction from the group $A_5 \times \mathbb{Z}/2$ to the subgroup D_5 (since A_5 is simple, the only possible non-trivial quotients would be isomorphic $\mathbb{Z}/2$, A_5, or $A_5 \times \mathbb{Z}/2$).

Let us start by observing that, from the inclusion $D_5 \hookrightarrow A_5$, we obtain that the inclusion $D_5 \hookrightarrow A_5 \times \mathbb{Z}/2$ factors through:

$$D_5 \hookrightarrow D_5 \times \mathbb{Z}/2 \cong D_{10} \hookrightarrow A_5 \times \mathbb{Z}/2$$

which implies the map on Whitehead groups likewise factors through:

$$Wh(D_5) \to Wh(D_{10}) \to Wh(A_5 \times \mathbb{Z}/2).$$

Observe that the first map in the above sequence was analyzed in the previous Section 7.2. Furthermore the last two groups in this sequence are abstractly isomorphic to $\mathbb{Z}^2$. So in order to obtain our claim, all we need to do is establish that the inclusion $D_{10} \hookrightarrow A_5 \times \mathbb{Z}/2$ induces an isomorphism on Whitehead groups.

In order to do this, we recall that Dress induction provides us with an isomorphism (see [O89, Chapter 11]):

$$Wh(A_5 \times \mathbb{Z}/2) \cong \varinjlim_{H \in \mathcal{H}(A_5 \times \mathbb{Z}/2)} Wh(H).$$

Here $\mathcal{H}(A_5 \times \mathbb{Z}/2)$ consists of all hyperelementary subgroups of $A_5 \times \mathbb{Z}/2$, the limit is over all maps induced by inclusion and conjugation, and the isomorphism is naturally induced by the inclusions. Now recall (Section 5.4) that the hyperelementary subgroups of $A_5 \times \mathbb{Z}/2$ are, up to isomorphism: $(\mathbb{Z}/2)^2$, $(\mathbb{Z}/2)^3$, D_3, D_5, D_6, and D_{10}. Amongst these groups (see Table 5), the only groups with non-trivial Wh are the groups D_5 and D_{10}, with $Wh(D_5) \cong \mathbb{Z}$ and $Wh(D_{10}) \cong \mathbb{Z}^2$. Furthermore, inside the group $A_5 \times \mathbb{Z}/2$, it is easy to see that:

1. every subgroup isomorphic to D_5 lies inside a subgroup isomorphic to D_{10},
2. all the subgroups isomorphic to D_{10} are pairwise conjugate.

This immediately implies that the direct limit to the right is canonically isomorphic to $Wh(D_{10})$, which gives us our desired claim.

Γ	$K_{-1} \neq 0$	$\tilde{K}_0 \neq 0$	$Wh \neq 0$
$[3,5,3]$	$\mathbb{Z}^4$	$\bigoplus_\infty \mathbb{Z}/2$	$\mathbb{Z}^3 \oplus \bigoplus_\infty \mathbb{Z}/2$
$[5,3,5]$	$\mathbb{Z}^4$	$\bigoplus_\infty \mathbb{Z}/2$	$\mathbb{Z}^6 \oplus \bigoplus_\infty \mathbb{Z}/2$
$[(3^3,4)]$	$\mathbb{Z}^2$	$(\mathbb{Z}/4)^2 \oplus \bigoplus_\infty \mathbb{Z}/2 \oplus Nil_0$	$\bigoplus_\infty \mathbb{Z}/2 \oplus Nil_1$
$[5,3^{1,1}]$	$\mathbb{Z}^2$	$\bigoplus_\infty \mathbb{Z}/2$	$\mathbb{Z}^3 \oplus \bigoplus_\infty \mathbb{Z}/2$
$[4,3,5]$	$\mathbb{Z}^3$	$(\mathbb{Z}/4)^2 \oplus \bigoplus_\infty \mathbb{Z}/2 \oplus Nil_0$	$\mathbb{Z}^3 \oplus \bigoplus_\infty \mathbb{Z}/2 \oplus Nil_1$
$[(3^3,5)]$	$\mathbb{Z}^2$	$\bigoplus_\infty \mathbb{Z}/2$	$\mathbb{Z}^3 \oplus \bigoplus_\infty \mathbb{Z}/2$
$[(3,5)^{[2]}]$	$\mathbb{Z}^4$	$\bigoplus_\infty \mathbb{Z}/2$	$\mathbb{Z}^6 \oplus \bigoplus_\infty \mathbb{Z}/2$
$[(3,4)^{[2]}]$	$\mathbb{Z}^4$	$(\mathbb{Z}/4)^4 \oplus \bigoplus_\infty \mathbb{Z}/2 \oplus Nil_0$	$\bigoplus_\infty \mathbb{Z}/2 \oplus Nil_1$
$[(3,4,3,5)]$	$\mathbb{Z}^4$	$(\mathbb{Z}/4)^2 \oplus \bigoplus_\infty \mathbb{Z}/2 \oplus Nil_0$	$\mathbb{Z}^3 \oplus \bigoplus_\infty \mathbb{Z}/2 \oplus Nil_1$

Table 6: The lower algebraic K-theory of the cocompact hyperbolic 3-simplex groups

7.4. **The spectral sequences.** By this point of the paper we have:

- described a simple model for $E_{\mathcal{FIN}}(\Gamma)$ for our groups, and identified the stabilizers of cells (in this section)
- computed (in Section 5) the lower algebraic K-groups of the stabilizers of the cells, and
- identified (in this section) the non-trivial morphisms appearing in the computation of the E^2-terms of the Quinn spectral sequence.

Furthermore, as explained earlier, the only possible non-zero terms in the spectral sequence are the $E_{p,q}$ with $p = 0, 1$. This boils down to understanding the homology of the complex:

$$0 \to \bigoplus_{\sigma^1} Wh_q(\Gamma_{\sigma^1}) \to \bigoplus_{\sigma^0} Wh_q(\Gamma_{\sigma^0}) \to 0,$$

But we've seen in this section that the middle map is always injective, hence the $E_{1,q}$ terms will also vanish. This gives us that *in all 32 cases the spectral sequence collapses at the E^2-term.* In fact, in all 32 cases, the only possible non-zero E^2-terms are $E^2_{0,-1}$, $E^2_{0,0}$, $E^2_{0,1}$. In particular the $K_i(\mathbb{Z}\Gamma)$ vanish for $i \leq -2$.

The results obtained for K_{-1}, $\tilde{K}_0$, and Wh for all 32 hyperbolic 3-simplex groups are listed out in Table 6 and Table 7. For ease of notation, we have only entered the non-zero terms in the Tables; all the blank squares represent entries where the corresponding group vanishes.

Γ	$K_{-1} \neq 0$	$\tilde{K}_0 \neq 0$	$Wh \neq 0$
$[3^{[\,]\times[\,]}]$			
$[4^{[4]}]$			
$[(3,6)^{[2]}]$	$\mathbb{Z}^2$		
$[6,3^{[3]}]$	$\mathbb{Z}^2$		
$[4^{1,1,1}]$			
$[(3,5,3,6)]$	$\mathbb{Z}^3$		$\mathbb{Z}^3$
$[(3,4^3)]$	$\mathbb{Z}^2$	$(\mathbb{Z}/4)^2$	
$[(3,4,3,6)]$	$\mathbb{Z}^3$	$(\mathbb{Z}/4)^2 \oplus Nil_0$	Nil_1
$[(3^3,6)]$	$\mathbb{Z}$		
$[3^{[3,3]}]$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$[6,3^{1,1}]$	$\mathbb{Z}$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$[3,6,3]$	$\mathbb{Z}^3$		
$[6,3,6]$	$\mathbb{Z}^6$	$(\mathbb{Z}/2)^4$	
$[4,4,4]$		$(\mathbb{Z}/4)^2 \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$[5,3^{[3]}]$	$\mathbb{Z}^2$	$\bigoplus_\infty \mathbb{Z}/2$	$\mathbb{Z}^3 \oplus \bigoplus_\infty \mathbb{Z}/2$
$[5,3,6]$	$\mathbb{Z}^5$	$\bigoplus_\infty \mathbb{Z}/2$	$\mathbb{Z}^3 \oplus \bigoplus_\infty \mathbb{Z}/2$
$[(3^2,4^2)]$	$\mathbb{Z}^2$	$(\mathbb{Z}/4)^2 \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$[4,3^{[3]}]$	$\mathbb{Z}^3$	$(\mathbb{Z}/4)^2 \oplus \bigoplus_\infty \mathbb{Z}/2 \oplus Nil_0$	$\bigoplus_\infty \mathbb{Z}/2 \oplus Nil_1$
$[3,3^{[3]}]$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$[3,4^{1,1}]$	$\mathbb{Z}^2$	$(\mathbb{Z}/4)^2 \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$[4,3,6]$	$\mathbb{Z}^4$	$(\mathbb{Z}/4)^2 \oplus \bigoplus_\infty \mathbb{Z}/2 \oplus Nil_0$	$\bigoplus_\infty \mathbb{Z}/2 \oplus Nil_1$
$[3,3,6]$	$\mathbb{Z}^4$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$[3,4,4]$	$\mathbb{Z}^2$	$(\mathbb{Z}/4)^2 \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$

Table 7: The lower algebraic K-theory of the non-cocompact hyperbolic 3-simplex groups

Note that several of the group appearing in Tables 6 and 7 involve copies of the Bass Nil-groups $NK_0(\mathbb{Z}D_4)$ and $NK_1(\mathbb{Z}D_4)$ (see Section 5.5). In order to simplify the notation in the tables, we will use Nil_0 and Nil_1 to denote these two Nil-groups. Recall that we know that the groups Nil_0, Nil_1 are torsion groups, where the order of every element divides 8, and furthermore the group Nil_0 is infinitely generated (see Section 5.5).

8. Appendix: two specific examples.

In this Appendix we work through the entire procedure for two specific examples (one cocompact, and one non-cocompact), with a view of helping the reader understand the layout of the paper.

8.1. **The group** $[(3,5)^{[2]}]$**.** The Coxeter diagram for this group Γ can be found in Figure 1, from which the following presentation can be read off (see Section 2):

$$\langle w, x, y, z \mid w^2 = x^2 = y^2 = z^2 = 1,$$
$$(wx)^3 = (xy)^5 = (yz)^3 = (zw)^5 = (wy)^2 = (xz)^2 = 1\rangle.$$

This group acts on $\mathbb{H}^3$ cocompactly, with fundamental domain a 3-simplex Δ^3. After labeling the hyperplanes extending the four faces by the four generators of Γ, the angles between these hyperplanes satisfy the following relationships (see Section 2):

- $\angle(P_w, P_y) = \angle(P_x, P_z) = \pi/2$,
- $\angle(P_w, P_x) = \angle(P_y, P_z) = \pi/3$,
- $\angle(P_w, P_z) = \angle(P_x, P_y) = \pi/5$.

In particular, the action of Γ on $\mathbb{H}^3$ gives a cocompact model for $E_{\mathcal{FIN}}(\Gamma)$, and the splitting formula (see Corollary 3.4) tells us that we have, for all $n \leq 1$, isomorphisms:

$$K_n(\mathbb{Z}\,\Gamma) \cong H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{i=1}^{k} H_n^{V_i}(E_{\mathcal{FIN}}(V_i) \to *).$$

Let us now identify the (finitely many) groups $\{V_i\}$ that appear in the above formula. As explained in Section 4, these groups will arise as stabilizers of Type I geodesics, which are precisely (up to the Γ-action) one of the six geodesics $P_w \cap P_x$, $P_w \cap P_y$, $P_w \cap P_z$, $P_x \cap P_y$, $P_x \cap P_z$, and $P_y \cap P_z$. To identify the stabilizers of these geodesics, we first need to identify the vertex stabilizers for the simplex Δ^3. Recall that these will be the special subgroups generated by triples of generators. But from the Coxeter diagram for Γ, one immediately sees that any triple of vertices spans out a subdiagram corresponding to the Coxeter group $[3,5]$. This implies that every vertex has stabilizer isomorphic to the (finite) Coxeter group $[3,5]$, which is well known to be isomorphic to the group $A_5 \times \mathbb{Z}/2$. Now for each of the six type I geodesics we have, one can consider the projection to the fundamental domain Δ^3. From Table 1, looking up the vertex stabilizers $A_5 \times \mathbb{Z}/2$, we see that every one of the six geodesics projects to precisely the associated edge in Δ^3. Now to find the stabilizers of the geodesics, one applies Bass-Serre theory. The stabilizer acts on each of the geodesics with quotient a segment, so one can write each of the stabilizers as a generalized free product. Furthermore, Table 1 allows us to identify the vertex groups in the Bass-Serre graph of groups.

Let us see how this works, for instance in the case of the geodesic $P_x \cap P_y$. The two associated hyperplanes P_x and P_y intersect at an angle of $\pi/5$, hence the edge group in the Bass-Serre graph of groups will be D_5. For the vertex groups, we see that the corresponding segment in Δ^3 joins a pair of vertices with stabilizer $A_5 \times \mathbb{Z}/2$, and correspond to the angle of $\pi/5$ at both the vertices. The last row in Table 1 tells us that both the vertex groups in the Bass-Serre graph of groups will be D_{10}. This tells us that the stabilizer of the geodesic $P_x \cap P_y$ is precisely the group $D_{10} *_{D_5} D_{10} \cong D_5 \times D_\infty$. Carrying this procedure out for each of the six geodesics, one finds that the stabilizers one obtains are:

- two copies of $D_{10} *_{D_5} D_{10}$, corresponding to the two geodesics $P_x \cap P_y$ and $P_w \cap P_z$,
- two copies of $D_6 *_{D_3} D_6$, corresponding to the two geodesics $P_w \cap P_x$ and $P_y \cap P_z$,
- two copies of $D_2 \times D_\infty$, corresponding to the two geodesics $P_w \cap P_y$ and $P_x \cap P_z$.

Note that these are precisely the groups that are listed out in Table 4. Finally, amongst these six subgroups, one needs to know which ones have a non-trivial cokernel for the relative assembly map. But from the work in Section 6, all the non-trivial cokernels are listed out in Table 6. Looking up Table 6, one sees that out of these six groups, the only ones with non-trivial cokernels are the two copies of $D_2 \times D_\infty$, each of whom contributes $\bigoplus_\infty \mathbb{Z}/2$ to the $K_0(\mathbb{Z}\Gamma)$ and $Wh(\Gamma)$.

So we are finally left with computing the homology coming from the finite subgroups, i.e. the term $H_n^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$. As we mentioned earlier, a cocompact fundamental domain for $\mathbb{H}^3/\Gamma$ is given by Δ^3. The stabilizers of cells in the fundamental domain can be read off from the Coxeter diagram, as they will precisely be the special subgroups (see the discussion in Section 7). We see that:

- there is one 3-dimensional cell (the interior of Δ^3), with trivial stabilizer,
- there are four 2-dimensional cells (the faces of Δ^3), with stabilizer $\mathbb{Z}/2$,
- there are six 1-dimensional cells (the edges of Δ^3), two of which have stabilizer D_2, two of which have stabilizer D_3, and two of which have stabilizer D_5,
- there are four 0-dimensional cells (the vertices of Δ^3), each of which has stabilizer $A_5 \times \mathbb{Z}/2$.

Now to obtain the E^2-terms in the Quinn spectral sequence, we need the homology of the complex:

$$\cdots \to \bigoplus_{\sigma^{p+1}} Wh_q(\Gamma_{\sigma^{p+1}}) \to \bigoplus_{\sigma^p} Wh_q(\Gamma_{\sigma^p}) \to \bigoplus_{\sigma^{p-1}} Wh_q(\Gamma_{\sigma^{p-1}}) \cdots \to \bigoplus_{\sigma^0} Wh_q(\Gamma_{\sigma^0}),$$

where σ^p are the p-dimensional cells (which we identified above). But from the work in Section 5, we know explicitly all the groups appearing in the above complex. Indeed, looking up the non-zero K-groups in Table 5, we see that for $q < -1$, the entire complex is identically zero. For the remaining values of q, we have:

$q = -1$: The complex degenerates to

$$0 \to 4K_{-1}(\mathbb{Z}[A_5 \times \mathbb{Z}/2]) \to 0,$$

where the four copies of $K_{-1}(\mathbb{Z}[A_5 \times \mathbb{Z}/2])$ come from the four vertices of Δ^3. Since we know (see Table 5) that $K_{-1}(\mathbb{Z}[A_5 \times \mathbb{Z}/2]) \cong \mathbb{Z}$, we immediately get that $E^2_{p,-1}$ all vanish, with the exception of $E^2_{0,-1} \cong \mathbb{Z}^4$.

$q = 0$**:** The complex is identically zero, and hence we see that $E^2_{p,0}$ all vanish.

$q = 1$**:** The complex degenerates to:

$$0 \to 2Wh(D_5) \to 4Wh(A_5 \times \mathbb{Z}/2) \to 0.$$

Note that the first copy of $Wh(D_5)$ comes from the edge $P_x \cap P_y \cap \Delta^3$, while the second copy of $Wh(D_5)$ comes from the $P_w \cap P_z \cap \Delta^3$. The four copies of $Wh(A_5 \times \mathbb{Z}/2)$ come from the four vertices of Δ^3.

Since the two edges $P_x \cap P_y \cap \Delta^3$ and $P_w \cap P_z \cap \Delta^3$ are disjoint, the complex splits as a sum of two subcomplexes, one for each of the two edges. Focusing on the first edge, we see that we have:

$$0 \to Wh(D_5) \to 2Wh(A_5 \times \mathbb{Z}/2) \to 0$$

We know that $Wh(D_5) \cong \mathbb{Z}$ and $Wh(A_5 \times \mathbb{Z}/2) \cong \mathbb{Z}^2$ (see Table 5), and that the map $Wh(D_5) \hookrightarrow Wh(A_5 \times \mathbb{Z}/2)$ induced by inclusion is split injective (see Section 7.3). This immediately tells us that in the chain complex above, we have that $2Wh(A_5 \times \mathbb{Z}/2)/Wh(D_5) \cong \mathbb{Z}^3$. An identical analysis for the other edge gives us that the homology of the original complex yields $E^2_{1,1} \cong 0$ and $E^2_{0,1} \cong \mathbb{Z}^6$.

Combining everything we've said so far, we see that for the Quinn spectral sequence, the only non-zero E^2-terms are $E^2_{0,-1} \cong \mathbb{Z}^4$ and $E^2_{0,1} \cong \mathbb{Z}^6$. This implies that the spectral sequence immediately collapses, giving us that

$$H^{\Gamma}_n(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong 0$$

for $n < -1, n = 0$, and

$$H^{\Gamma}_{-1}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong \mathbb{Z}^4,$$
$$H^{\Gamma}_{1}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong \mathbb{Z}^6\,.$$

We now have both the terms appearing in the splitting formula, and we conclude that the lower algebraic K-theory of the group Γ is given by:

$$Wh_n(\Gamma) = \begin{cases} Wh(\Gamma) \cong \mathbb{Z}^6 \oplus \bigoplus_\infty \mathbb{Z}/2, & n = 1 \\ \tilde{K}_0(\mathbb{Z}\Gamma) \cong \bigoplus_\infty \mathbb{Z}/2, & n = 0 \\ K_{-1}(\mathbb{Z}\Gamma) \cong \mathbb{Z}^4, & n = -1 \\ K_n(\mathbb{Z}\Gamma) \cong 0, & n \leq -1. \end{cases}$$

Looking up Table 6, one finds that these are precisely the values reported.

8.2. **The group** $[3, 4^{1,1}]$**.** The Coxeter diagram for this group Γ can be found in Figure 2, from which the following presentation can be read off (see Section 2):

$$\langle w, x, y, z \mid w^2 = x^2 = y^2 = z^2 = 1,$$
$$(wx)^3 = (xy)^4 = (yz)^2 = (zw)^2 = (wy)^2 = (xz)^4 = 1\rangle.$$

This group acts on $\mathbb{H}^3$ with cofinite volume, with fundamental domain a (non-compact) 3-simplex Δ^3 with one ideal vertex. After labeling the hyperplanes extending the four faces by the four generators of Γ, the angles between these hyperplanes satisfy the following relationships (see Section 2):

- $\angle(P_w, P_y) = \angle(P_w, P_z) = \angle(P_w, P_z) = \pi/2$,
- $\angle(P_w, P_x) = \pi/3$,
- $\angle(P_x, P_z) = \angle(P_x, P_y) = \pi/4$.

The ideal vertex arises as the intersection (at infinity) of the three hyperplanes $P_x \cap P_y \cap P_z$, and has stabilizer the 2-dimensional crystallographic group $[4, 4]$.

While the action of Γ on $\mathbb{H}^3$ does not give a cocompact model for $E_{\mathcal{FIN}}(\Gamma)$, one can obtain such a model by Γ-equivariantly truncating disjoint horospheres centered at the Γ-orbits of the ideal vertex (see Section 7). The splitting formula (see Corollary 3.4) tells us that we have, for all $n \leq 1$, isomorphisms:

$$K_n(\mathbb{Z}\,\Gamma) \cong H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{i=1}^{k} H_n^{V_i}(E_{\mathcal{FIN}}(V_i) \to *).$$

Let us now identify the (finitely many) groups $\{V_i\}$ that appear in the above formula. As explained in Section 4, these groups will arise as stabilizers of Type I geodesics, which are precisely (up to the Γ-action) one of the six geodesics $P_w \cap P_x$, $P_w \cap P_y$, $P_w \cap P_z$, $P_x \cap P_y$, $P_x \cap P_z$, and $P_y \cap P_z$. Note that since the geodesic segments $P_x \cap P_y$, $P_y \cap P_z$ and $P_x \cap P_z$ project to non-compact segments in the fundamental domain (they give rise to edges joined to the ideal vertex), these geodesics will *never* have an infinite stabilizer, and we can hence safely ignore them.

To identify the stabilizers of the remaining three geodesics, we follow the procedure from Section 4. We first need to identify the (non-ideal) vertex stabilizers for the simplex Δ^3. Recall that these will be the special subgroups generated by triples of generators. But from the Coxeter diagram for Γ, one immediately sees that the triple of vertices span out the subdiagrams:

- the Coxeter group $[3, 4] \cong S_4 \times \mathbb{Z}/2$ will be the stabilizer of the vertices $P_w \cap P_x \cap P_z$ and of the vertex $P_w \cap P_x \cap P_y$,
- the group $(\mathbb{Z}/2)^3$ will be the stabilizer of the vertex $P_w \cap P_y \cap P_z$.

Now for each of the three (potentially cocompact) type I geodesics that we have ($P_w \cap P_x$, $P_w \cap P_y$, and $P_w \cap P_z$) one can consider the projection to the fundamental domain Δ^3. From Table 1, looking up the vertex stabilizers $S_4 \times \mathbb{Z}/2$, we see that every one of the three geodesics projects to precisely the associated edge in Δ^3.

To find the stabilizers of these geodesics, we now use Bass-Serre theory as explained in Section 4. To find the vertex groups, one uses Table 1, while the edge group will be precisely the dihedral group given by the special subgroup associated to the geodesic. This immediately gives us the stabilizers:

- one copy of $D_6 *_{D_3} D_6$, corresponding to the geodesic $P_w \cap P_x$,
- two copies of $(\mathbb{Z}/2 \times D_2) *_{D_2} (\mathbb{Z}/2 \times D_2) \cong D_2 \times D_\infty$, corresponding to the two geodesics $P_w \cap P_y$ and $P_w \cap P_z$,

which are precisely the groups reported in Table 3. Finally, amongst these three subgroups, one needs to decide which ones have a non-trivial cokernel for the relative assembly map. These cokernels are listed out in Table 6, and one sees that the only non-trivial contribution will come from the two copies of $D_2 \times D_\infty$, each of which will contribute $\bigoplus_\infty \mathbb{Z}/2$ to the $\tilde{K}_0(\mathbb{Z}\,\Gamma)$ and $Wh(\Gamma)$.

So we are finally left with computing the homology coming from the finite subgroups, i.e. the term $H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$. As we mentioned earlier, a cocompact fundamental domain for $\mathbb{H}^3/\Gamma$ is given by "truncating" the ideal vertex from Δ^3.

The stabilizers of cells in the fundamental domain can be read off from the Coxeter diagram, as they will precisely be the special subgroups (see the discussion in Section 7). We see that:

- there is one 3-dimensional cell (the interior of Δ^3), with trivial stabilizer,
- there are five 2-dimensional cells, with stabilizer $\mathbb{Z}/2$ (for the faces of the original Δ^3), or trivial (for the face coming from truncating the ideal vertex in Δ^3),
- there are nine 1-dimensional cells (the six edges of Δ^3, and three edges obtained from the truncation). Three of these will have stabilizer $\mathbb{Z}/2$ (those coming from truncating the ideal vertex in Δ^3), three will have stabilizer D_2 (from the edges corresponding to $P_y \cap P_z$, $P_w \cap P_y$, and $P_w \cap P_z$), two will have stabilizer D_4 (from the edges corresponding to $P_x \cap P_y$ and $P_x \cap P_z$), and one with stabilizer D_3 (from the edge corresponding to $P_w \cap P_x$),
- there are six 0-dimensional cells (three non-ideal vertices of Δ^3, and three from the truncation of the ideal vertex). Two have stabilizers D_4 (from the truncation of the two edges with the same stabilizer), one has stabilizer D_2 (from the truncation of the third edge), two have stabilizer $S_4 \times \mathbb{Z}/2$ (from two of the non-ideal vertices), and one has stabilizer $(\mathbb{Z}/2)^3$ (from the third non-ideal vertex).

Now to obtain the E^2-terms in the Quinn spectral sequence, we need the homology of the complex:

$$\cdots \to \bigoplus_{\sigma^{p+1}} Wh_q(\Gamma_{\sigma^{p+1}}) \to \bigoplus_{\sigma^p} Wh_q(\Gamma_{\sigma^p}) \to \bigoplus_{\sigma^{p-1}} Wh_q(\Gamma_{\sigma^{p-1}}) \cdots \to \bigoplus_{\sigma^0} Wh_q(\Gamma_{\sigma^0}),$$

where σ^p are the p-dimensional cells (which we identified above). But from the work in Section 5, we know explicitly all the groups appearing in the above complex. Indeed, looking up the non-zero K-groups in Table 5, we see that *the only one of the cell stabilizers that has non-trivial K-theory is the group $S_4 \times \mathbb{Z}/2$*. There are two copies of this group, arising as stabilizers of 0-cells, and we have that $K_{-1}(\mathbb{Z}[S_4 \times \mathbb{Z}/2]) \cong \mathbb{Z}$ and $\tilde{K}_0(\mathbb{Z}[S_4 \times \mathbb{Z}/2]) \cong \mathbb{Z}/4$. This immediately tells us that non-zero terms in the Quinn spectral sequence will be $E^2_{0,-1} \cong \mathbb{Z}^2$ and $E^2_{0,0} \cong (\mathbb{Z}/4)^2$. This implies that the spectral sequence immediately collapses, giving us that

$$H_n^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong 0$$

for $n < -1, n = 1$, and

$$H_{-1}^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong \mathbb{Z}^2,$$
$$H_0^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong (\mathbb{Z}/4)^2.$$

We now have both the terms appearing in the splitting formula, and we conclude that the lower algebraic K-theory of the group Γ is given by:

$$Wh_n(\Gamma) = \begin{cases} Wh(\Gamma) \cong \bigoplus_\infty \mathbb{Z}/2, & n = 1 \\ \tilde{K}_0(\mathbb{Z}\Gamma) \cong (\mathbb{Z}/4)^2 \oplus \bigoplus_\infty \mathbb{Z}/2, & n = 0 \\ K_{-1}(\mathbb{Z}\Gamma) \cong \mathbb{Z}^2, & n = -1 \\ K_n(\mathbb{Z}\Gamma) \cong 0, & n \leq -1. \end{cases}$$

Looking up Table 7, one finds that these are precisely the values reported.

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